On the Geometry and Kinematics of (Semi-) Rigid Bodies.

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June 09, 2022

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Standard Texts – Modeling and Control

Robot Modeling and Control

Spong, Mark W., Seth Hutchinson, and Mathukumalli Vidyasagar. Robot modeling and control. Vol. 3. New York: Wiley, 2006.

Mathematical Modeling of Robots

Murray, R. M., Li, Z., & Sastry, S. S. (1994). A Mathematical Introduction to Robotic Manipulation. In Book (Vol. 29). https://doi.org/10.1.1.169.3957

Texts – Modeling, Control, and Mechanisms

Robot Modeling and Control

Lynch, K. M., & Park, F. C. (2017). Modern Robotics Mechanics, Planning, and Control.

Mechanisms' Kinematic Geometry

Hunt, Kenneth H., and Kenneth Henderson Hunt. Kinematic geometry of mechanisms. Vol. 7. Oxford University Press, USA, 1978.

Texts – Screws and Kinematics

Screw Theory

Ball, Robert Stawell. A Treatise on the Theory of Screws. Cambridge university press, 1998.

Mechanisms' Kinematic Geometry

Hunt, K. H. (2019). Structural Kinematics of In-Parallel-Actuated Robot-Arms. 105(December 1983), 705–712.

Lecture One Outline

Mechanism Components

Kinematic geometry. Mechanisms.

Joints: Joint closure; Pairs; Couplings.

Lower pairs and linkages; Higher and lower pairs.

Motions: Planar and spherical motions.

Synthesis: Type-, number-, and size-syntheses.

Preamble.

Mechanics

Mechanics is an indirect study of nature via bodies – essentially the mathematical abstractions of common natural things; the mass is an *allocation* in *place* to each body.

Geometry

Geometry, deals with the theory of places; geometry is the bedrock of robotics, control theory, and many fields of modern engineering and the physical sciences.

Mechanics Overview.

Definition (Motion)

When a place undergoes body transformation in the course of time, we have motion.

Preamble – Mass, Body, Rigid Body Motion.

Definition (Body – Truesdell, 1977.)

By a body, we shall mean the closure of an open set in some measure space Ω over which a non-negative measure M, called the mass, is defined, and that M can be extended to a Borel measure over the $\sigma-$ algebra of Borel sets in Ω.

Bodies – Truesdell, 1977.

That in mechanics which deals with
(i) mass points, which occupy a single point at any one time;
(ii) rigid bodies, which never deform;
(iii) strings and rods and jets, which are 1-dimensional; membranes and shells, that sweep out surfaces;
(iv) space-filling fluids and solids e.t.c. are termed bodies.

Statics, Dynamics, Rigid Body (Motion).

Statics and Dynamics

That which studies putative equilibria is referred to as statics. That which concerns motion of all sorts is referred to as dynamics. The dynamics that are specific to particular bodies are termed constitutive.

Statics, Dynamics, Rigid Body (Motion).

The Rigid Body

A rigid body does not stretch, buckle, contract, bend, twist, nor deform. Well, not really!

The Rigid Body

As engineers and roboticists, we judge kinematic rigid hardware with the expectation that kinematic changes do not depart from rigid-body predictions.

Statics, Dynamics, Rigid Body (Motion).

The Rigid Body

We expect that localized stresses, active noise, vibrations and heat e.t.c will not cause reasonable departures from expectations.

Rigid Body Motion

That motion that preserves distance between all points in a body is termed a rigid body motion.

Statics, Dynamics, Rigid Body (Motion).

Rigid Body Motion

At issue are components of a rigid body's movement w.r.t to a fixed or moving frame of reference. In its most basic form, this movement is parameterized by displacement (and is sometimes time-varying e.g. for a continuum body). When solving for the movements of bodies, it is often useful to include velocities (twists) in order to characterize the motion.

Kinematics vs. Kinetics

Dynamics

$$\dot{x} = f(t; x, u), \quad x(t_0) = t_0 \tag{1}$$

$$\dot{x} = f(t; x) + g(t; x, u), \quad x(t_0) = t_0 \tag{2}$$

Definition (Kinematics.)

Kinematics is the English version of the word *cinématique* coined by A.M. Ampère (1775-1836), who translated it from the Greek word *κίνημα*.

Kinematics vs. Kinetics.

Definition (Truesdell)

That part of a system's dynamics that involves its motion by displacement – both linear and angular – and separated from motions owing to forces and torques, together with the successive derivatives with respect to time of all such displacements (this includes velocities, accelerations, and hyper accelerations) all form the kinematics of a rigid, continuum or laminae of bodies.

Kinematics vs. Kinetics

Kinetics

The motion of bodies can also be conceived as resulting from the forces' action. Energy, temperature, and calory of a body are resultant effects of gains or loss of heat. Motions arising as a result of these are called kinetics.

Kinematics vs. Kinetics.

Definition (Kinetics – Technical Definition)

That part of a system's dynamics that involves its motion by forces, energy, torque, inertia, dynamic stability, and equilibrium and similar properties all form the kinetics of a rigid, continuum or laminae of bodies.

Definition (Kinematic Geometry)

The solid geometry of relatively moving rigid bodies is termed the kinematic geometry of the rigid body. With motion, we'd have to include the successive derivatives of the displacement such as acceleration e.t.c as the 'laws of motion' stipulates in mechanics.

Joints and Links

Links

Links may be rigid mechanical parts, elastic, (vulcanized) rubber components, diaphragms, conveyor belts, spring-damper systems e.t.c.

An Elementary Joint or Kinematic Pair.

An elementary joint or a kinematic pair consists of touching two links together at one point – then ensuring a single contact point is continuously maintained throughout relative movement.

Joint (Contact) Kinematics

Contact Kinematics

A body may slide or slip across a plane or surface, or roll over another body.

Joints

Joints are the result of the connecting points between two or more rigid bodies.

Definition of a Mechanism

Definition (Author's Definition)

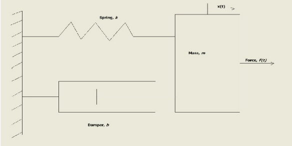
A connection of mechanical, magnetic, electrical, hydraulic, or pneumatic components forming an assemblage, meant for moving rigid, semi-rigid or non-rigid bodies via a controlled generation of (sometimes constrained) motion.

Kenneth Hunt (1978)

A means of transmitting, controlling, or constraining the relative movement between parts. Whenever we have an higher pair or more, we have a mechanism.

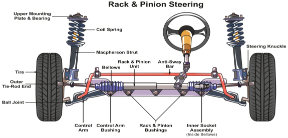
Mechanism Examples

Spring-Mass-Damper System

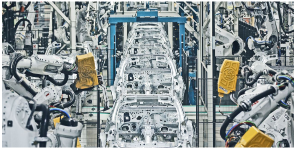


Excavator

Car suspension



Daimler Plant



Lower Pairs, Higher Pairs, Linkages

Lower and Higher Pairs

When elements of pairs touch one another over a substantial region of a surface covering a line, curve-surface, or point of contact, we have lower pairs. When they touch along a discrete line, curve-surface, or point of contact, we have higher pairs.

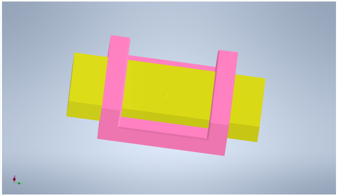
Linkage (Hunt, 1978)

If all joints of a mechanism or mechanical movement belong to lower pairs, we have a linkage.

Prismatic Pairs or P-pairs

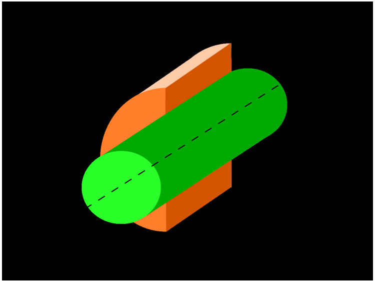
Hunt, 1978

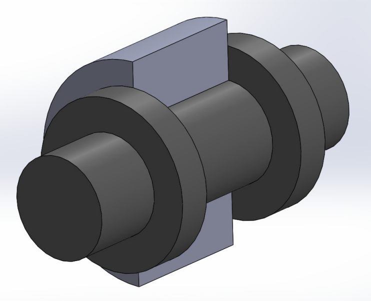
Formed by receding the axis of the revolution surface between two pairs to ∞ so that the curve that produces the surface moves parallel to itself, tracing a cylinder; or a polygonal-tracing curve generates a prism.



Revolute Pairs or R-pairs

One convex surface and one non-convex surface for a one degree of rotational freedom around the one joint the two surfaces make.





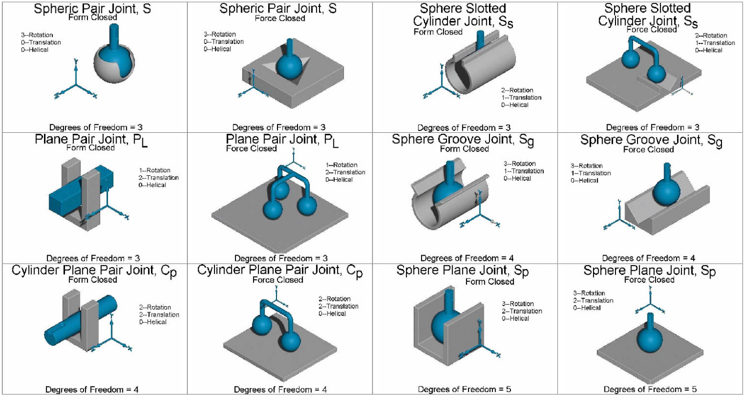
Revolute or Hinge or Turning or simply R-pairs with and without shoulder cutaway geometries. Credit: Wikimedia commons.

Helical- & U-Joints

Helical Joint

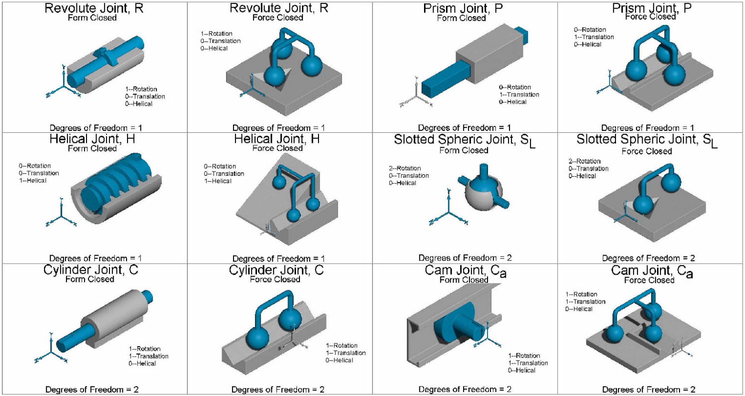
Universal Joint

Common Lower Kinematic Pairs



Credit: Wharton and Singh, 2001.

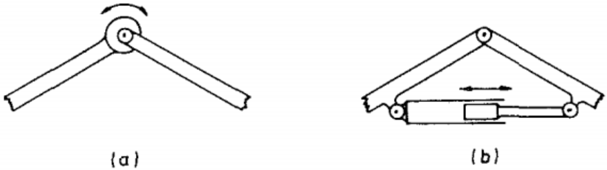
Common Lower Kinematic Pairs



Credit: Wharton and Singh, 2001.

Kinematic Geometry of Common Actuations

In-series vs. Parallel-actuated lower pairs



(a): In-series-actuated kinematic pair with a rotary joint that is actuated "about" the hinge. (b): Prismatic joint actuated "across" a hinge. Reprinted from Hunt, Kenneth. Structural Kinematics of In-Parallel-Actuated Robot Arms. Transactions of ASME. 1983.

Kinematic Chains

Kinematic Chains (Reuleaux, 1975)

We can explain the structural similarity of many mechanisms by parts of kinematic chains connected by pairs.

Kinematic chains

Kinematic chains are essentially the basic building structure of mechanisms . . . and robots!

Open Kinematic Chains

Chains

Open kinematic chains are based off the anthropomorphic construction of the human hand with cantilevered beam structures.

Chain Mechanisms and Error Amplification

Amplifies errors from waist (or base frame) all the way to the tool frame. Control difficult.

Control

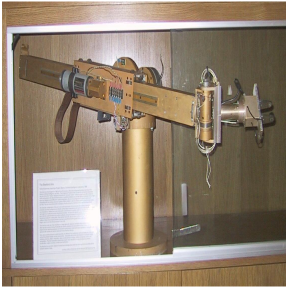
Feedforward control: High power and precision hydraulic actuators for servo motors.
Sensory feedback control: Force sensing (Ernst, 1962).

Open Kinematic Lower Pairs

Definition (Ken Salisbury Jr., 1982)

"[Robots are] our fascination with constructing mechanical analogues of ourselves... [this fascination] has led us to place all sorts of hopes and expectations in robot capabilities."

The PUMA Robot (1956).



The Stanford Arm (Infolab 1969).

Open Kinematic Chains

Open kinematic chains provide unstructured environmental interaction.

Project MAC, MIT.

Tomovic and Boni's pressure sensed grasp.

Binary robot vision system (McCarthy et al, 1963).

Open Kinematic Chains

Stanford Manipulator.

Boston arm.

The AMF (American Machines and Foundry) arm.

General electric's walking robot (1969).

Long Walk Towards Direct Drive Robot Arms

The 50's, 60's nd 70's witnessed use of hydraulics for (feedforward) position control.

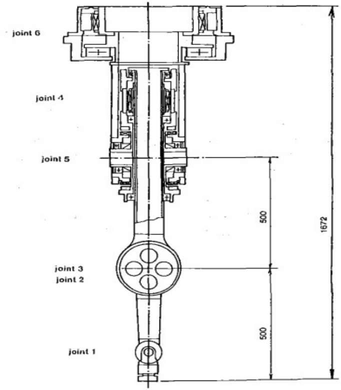
For feedback control, force sensors and pressure sensors were used in closed-loop scenarios.

Electrical actuation meant that robots had to be operated at high speeds. Needs for gear reduction for safe operations at low speeds.

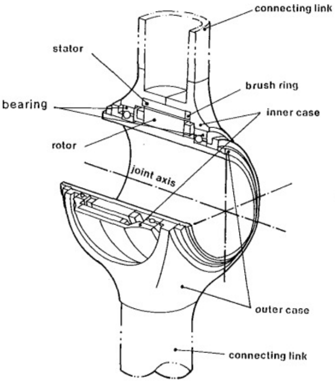
With gear reduction came backlash, friction, and associated expenses.

Direct Drive Robot Mechanism: CMU DD I Arm

Along came Harry Asada.

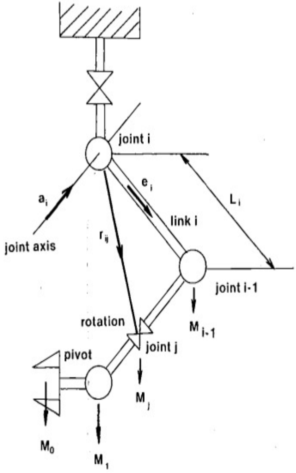


Arm Schematics Transmission

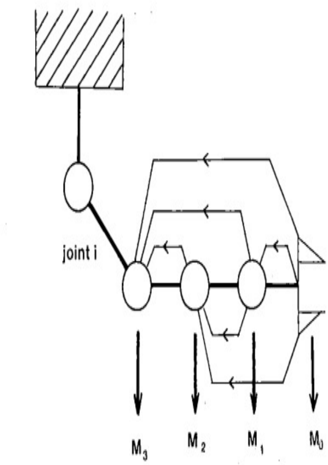


Joint schematic

Direct Drive Robot Mechanism: CMU DD I Arm



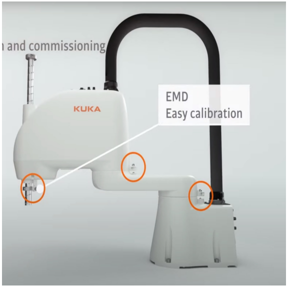
Kinematic model



Errors Transmission

SCARA Robot Mechanisms

The Adept One SCARA robot (Debuted 1984).



Kuka's SCARA arm, 2022. ©Kuka Robotics

The Stäubli anthropomorphic arm.

Serial mechanisms research in the 80's

Mechanisms in the 80's

With the 80's came the arrival of PCs. Lots of research went into computational algorithms for the kinematics and kinetics of (mostly) anthropomorphic robot arms.

Active control schemes

Efficient recursive Lagrangian and computational methods for the gravitational and Coriolis forces in Newton-Euler equations.

Serial mechanisms research in the 80's

Feedback Linearization

Dynamics feedback linearization for precise bounds on manipulator performance.

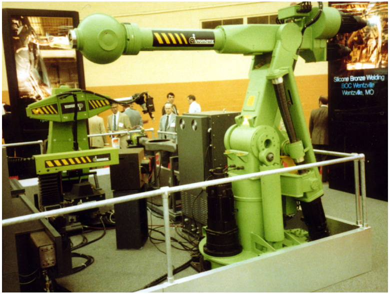
Automatix

Reconfigurable robots for various assembly ops.

Serial mechanisms research in the 90's

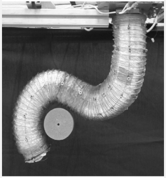
Robotworld

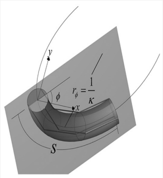
First industrial-scale reconfigurable robot and with machine vision components. RAIL scripting OS originally based on Motorola 68000, later on replaced by Apple Macintosh II.

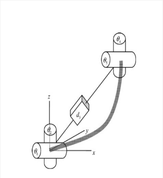


©Wikipedia

Hyper-redundant Continuum Robots

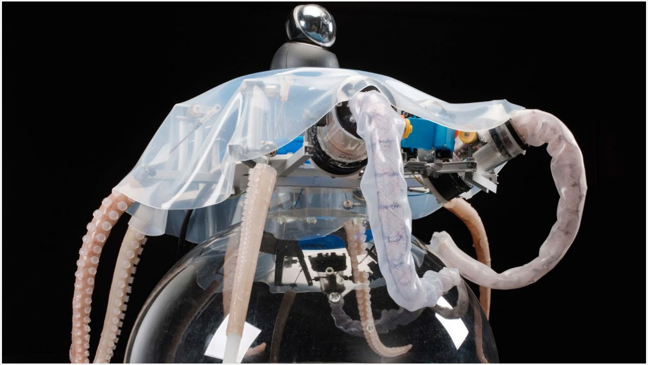






The elephant trunk continuum robot. Jones & Walker, T-RO 2006.
Inspiration: Muscular hydrostats in nature.

Hyper-redundant Kinematic Chains



An octopus-inspired soft robot. ©Cecilia Laschi.

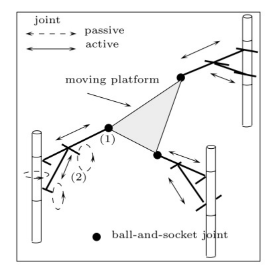
Parallel Robots

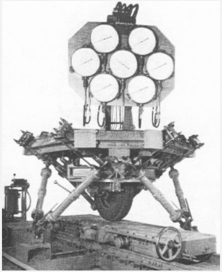
Mehlet 2015

A parallel robot is made up of an end-effector with n degrees of freedom, and of a fixed base, linked together by at least two independent kinematic chains. Actuation takes place through n simple actuators.

Parallel mechanisms: Stewart-Gough Platforms

Principles of a moving platform to test tyre wear and tear (Gough, 1947). Prototype, 1955.





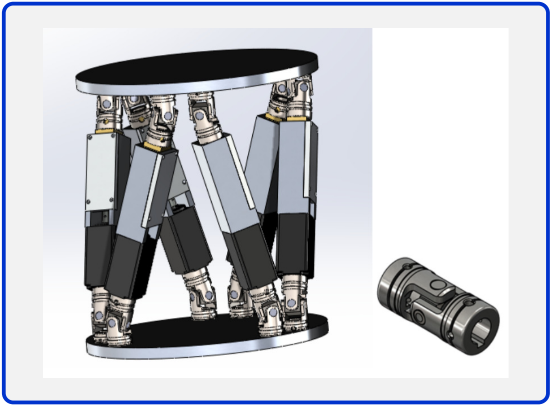
Left: Stewart's 1965 mechanism. *Right:* The original 1954 octahedral hexapod proposed by Gough. Courtesy: Parallemic.org.

Truss Robots

A multi-DOF Truss Robot. Courtesy of Penngineering (ICRA 2022, Philadelphia, PA).

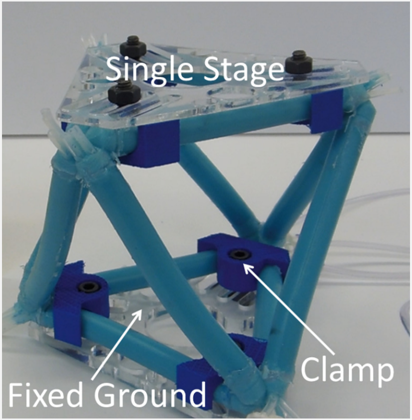
Closed kinematic chains

Connection degree ≥ 3.



A Stewart-Gough platform. SolidWorks Drawing Courtesy of Andrew Belcher. UChicago, 2018.

A Soft Stewart Platform



A soft 6-6 Stewart manipulator. Jonathan Hopkins, 2015.

Lecture II Outline

Freedom and Structure

Freedoms, Constraints, and Mobility.

Motion of linkages: Screws and spatial motions.

Freedom and Mobility: Freedoms, unfreedoms, connectivity, mobility;

Grübler-Kutzbach's mobility criterion and examples;

Type-, size- and number-syntheses.

Degrees of Freedom and Kinematic Structure

Definition (Connection Degree)

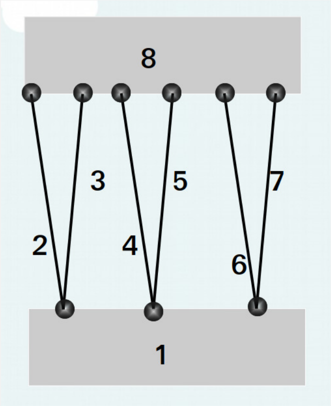
For any manipulator joint, we shall mean its connection degree to be the number of links attached it.

Quiz

What is the connection degree of the u-joints of a Stewart-Gough platform.

Members and Dual Graphs

Dual graph of a Stewart platform



Degrees of Freedom and Structure

Members and Freedoms

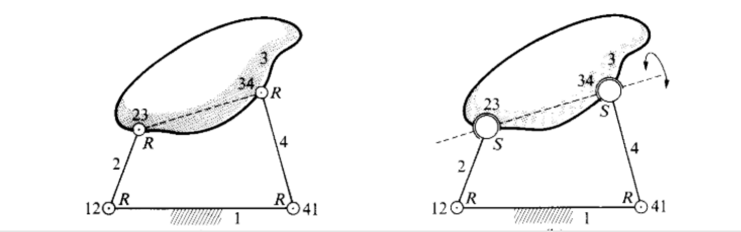
Degrees of freedoms (or freedoms) concerns the relative motion of members of a pair that do not touch one another directly.

Connectivity

By the dual graph of the Stewart platform as seen on Frame 54, the total number of freedoms that connect the two members (1 and 8) that do not connect to one another directly is six.

Planar Linkages

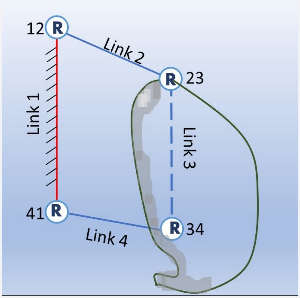
Four Bar Linkages



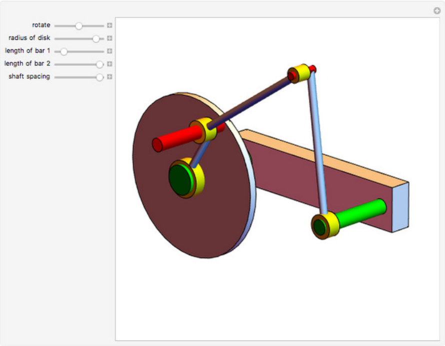
Reprinted from Hunt, 1977: Kinematic Geometry of Mechanisms.

Freedom from Connectivity

A (Hacked) Four-Bar Linkage



A Four Bar Linkage



Courtesy of Sándor Kabai, Wolfram Demonstrations Project, October 2007.

Couplings and Freedom

Couplings and Freedom

Links $2\&4$ complete a coupling or connection between links $1\&3$.

Connectivity

The R-pairs are said to have a connectivity of $\mathscr{C}_{ij} = 1$ for all $i, j = 1, 2, 3, 4$. Thus, total degree of freedom is 1.

Mobility of Mechanisms

The Mobility and Relative Mobility, $\mathfrak{M}$

Simply put, the number of a mechanism's freedoms is its mobility, or relative mobility, $\mathfrak{M}$.

The Mobility, $\mathfrak{M}$

It specifies the independent variables needed to determine every relative location of a mechanism's members with respect to one another.

A Note on Serial and Parallel Mobility

A little tricky to determine for parallel mechanisms but straightforward for serial mechanisms.

Mobility of Mechanisms

Quiz

What is the mobility $\mathfrak{M}$ of the $RSSR$ four bar linkage of Frame 56? Why?

Quiz

What is the mobility $\mathfrak{M}$ of the $RRRR$ four bar linkage of Frame 57? Why?

Definition (The mobility criterion (well, not yet))

Let's not get ahead of ourselves. A little introduction to screws are in order for us to grasp the Grübler-Kutzbach mobility criterion.

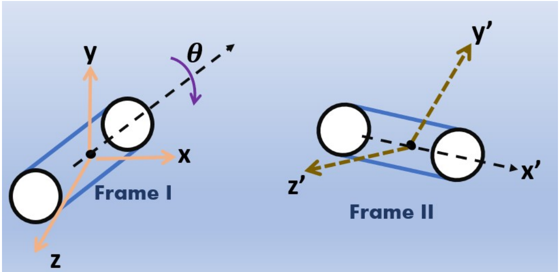
Inhomogeneity of Displacements and Angles

Quiz: Three translations and three rotations are ill-posed for uniquely determining the freedoms of a body. Why?

They are not homogeneous.

For true kinematic wholeness and generality, displacement that is purely translatory and purely rotary is needed.

Need for Unique Representations



There exists infinite possible ways of movement by which the point on the rigid body in Frame I can be effected to be transferred to the location of the point on Frame II and vice versa.

Screws for Kinematic Generality

Need for Screws

From a kinematic standpoint, six homogeneous screw coordinates – each having an independent screw freedom – are needed to uniquely determine a rigid body's location.

Definition (What is a screw anyway?)

A screw is a straight line in space, called the axis, with an associated direction, called pitch, p.

Screws in Mechanics: A History Snippet

Screws Through Time

Michel Chasles(1793 – 1880): Rigid Body Displacements (Kinematics).

Louis Poinsot (1777 – 1859): Geometrical Mechanics (Kinetics).

Sir Robert Stawell Ball, F.R.S, LL.D. (1840-1913): Irish Astronomer who popularized screw theory in his day.

Screws in Mechanics: A History Snippet

Screws Through Time

The Death of Screw Theory

After World War I, interest in Screw theory declined. Several possible reasons for this.

- Ball died in 1913, he had no students.
- Other British/Irish mathematicians who might have carried these ideas forward died young,
  - Clifford, a contemporary and friend of Ball's died 1879 aged 33.
  - Charles Jasper Joly (1864-1906) studied under Helmholz and Königs in Berlin, was a successor of Ball's as Royal Astronomer of Ireland.
  - Arthur Buchheim (1859–1888), studied under Klein at Leipzig. Taught at Manchester grammar school, died aged 29.
- Relativity became popular, and Euclidean geometry less so. Ball joked, "The Theory of Screws is now all done; it is quite obsolete; it is all going over into non-Euclidean space."

Kept alive in Soviet Union, Kotelnikov and others.

Rediscovered by Mechanical Engineers

In the 1960s two mechanical engineers in Australia rediscovered Balls work. Ball's theory of screws was just what they needed to study mechanisms.

- Kenneth Henderson Hunt (1920–2002) was born in the UK, worked at Monash University. "Kinematic Geometry of Mechanisms", first published in 1978. Applied screw theory to the problem of designing constant velocity joints.
- Jack Raymond Phillips (1923–2009) University of Sydney. Studied agricultural machinery, (trailed disc ploughs) and the mechanics of the lobster's claw. Two volume work "Freedom in Machinery: Introducing Screw Theory".

Courtesy of J.M. Selig, IROS 2018 Screw Theory Tutorial.

Screws for Kinematic Generality

Simplest (Unique) Representation of Displacements

Chasles(1793 – 1880) showed that any given displacement of a rigid body can be uniquely represented as the rotation of the body about an axis (the screw axis) followed by a translation parallel to that axis (the pitch).

Michel Chasles and Screws

Chasles called this unique transformation screws. Chasles is responsible for the Euclidean description of the motion of a rigid body in space and he made lasting contributions to theories of rigid body dynamics.

Screws for Kinetic Generality

Simplest (Unique) Representation of Forces

Poinsot(1777 – 1859) showed that any system of forces acting on a rigid body can be represented by a single force, together with a couple acting along the normal to the force in a plane.

Louis Poinsot and Geometrical Mechanics

"Everyone makes for himself a clear idea of the motion of a point, that is to say, of the motion of a corpuscle which one supposes to be infinitely small, and which one reduces by thought in some way to a mathematical point." $\sim$ Louis Poinsot, 1834.

Freedom and Constraints

Suppose a screw $\boldsymbol{f} = (f_1, \cdots, f_6)$ "fixes" a body in 3D space.

Each constraint $u_i \neq f_j$ for $(i, j) \in \{1, \cdots, 6\}$.

Rather each u_i has influence on every $\{f_i\}_{i=1}^6$.

Each u_i from the six independent equations, $g(s_1, s_2, s_3, s_4, s_5, s_6) = 0$, suppresses a freedom, f_i.

Progressively relaxing each u_i, or unfreedom, adds an extra body f_i.

Freedoms, Unfreedoms, and Mobility

Freedom and Unfreedoms

Suppose the total freedoms is $\boldsymbol{f}$ and the total unfreedoms is $\boldsymbol{u}$, then

$$\boldsymbol{u} + \boldsymbol{f} = 6.$$

Note: A rigid body's freedoms is also referred to the dimension of its configuration space.

Relative Freedoms

Suppose there are a total of n unconstrained bodies. Suppose further that we choose one out of the bodies as a reference body. Then the total number of relative freedoms is $6(n-1)$.

Freedoms, Unfreedoms, and Mobility

Constraints and Joints

Now, consider k independent constraints[a] such as joints along points, lines, curves or surfaces.

[a] NB: The total *allowable* constraints is 5 for a body in relative motion. I 6 for a fully rigid body.

The Mobility Criterion

Let the constraint of joint, i (e.g. a joint along points, lines, curves or surfaces) be u_i. Then the mobility criterion $\mathfrak{M}$ is

$$\mathfrak{M} = 6(n-1) - \sum_{i=1}^{k} u_i. \tag{3}$$

General Grübler-Kutzbach Mobility Criterion

Recall that $\sum_i u_i + f_i = 6$ from Frame (70) so that

$$\mathfrak{M} = 6(n - k - 1) - \sum_{i=1}^{f} f_i. \tag{4}$$

Exceptions: Relative Planar and Spherical Motions

For bodies restricted to relative planar or spherical motions, the total freedoms + constraints is 3 (not 6)!

$$\mathfrak{M} = 3(n - k - 1) - \sum_{i=1}^{f} f_i. \tag{5}$$

General Grübler-Kutzbach Criterion References

The Grübler-Kutzbach Mobility Criterion References

Attributed to Grübler:

Schoenflies, Arthur, and M. Grübler. "Kinematik." In Encyklopädie der Mathematischen Wissenschaften mit Einschluss ihrer Anwendungen, pp. 190-278. Vieweg+Teubner Verlag, Wiesbaden, 1908;

Grübler, Martin Fürchtegott. Getriebelehre: eine Theorie des Zwanglaufes und der ebenen Mechanismen. Springer, 1917.

and Kutzbach:

Kutzbach, Karl. "Mechanische leitungsverzweigung, ihre gesetze und anwendungen." Maschinenbau 8, no. 21 (1929): 710-716.

Loops

Loops

A kinematic chain often comprises members called loops.

Binary Link

Members in a binary link constitute a single loop. Example: The four-bar linkage.

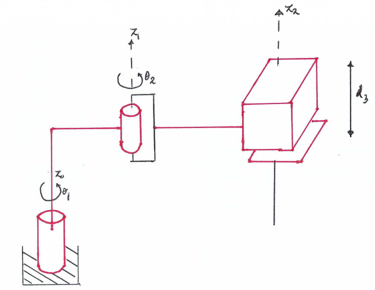
Single loops

For single loops, $k = n$ so that $\mathfrak{M} = \sum_{i=1}^{f} f_i - 6$.

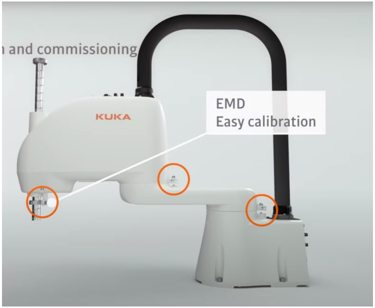
Mobility of Mechanisms

$\mathfrak{M} \leq 1$ for at least one actuator-pair to produce mobility at a successor joint which depends on that actuator-pair's input.

Mobility of Common Robot Configurations



Configuration of the SCARA Arm.



Courtesy of Fanuc America Inc.

Mobility of The SCARA Robot

Mobility Analysis

Two rotary joints. One prismatic joint acting along the z axis, and constrained along the xy plane.

Mobility Parameters

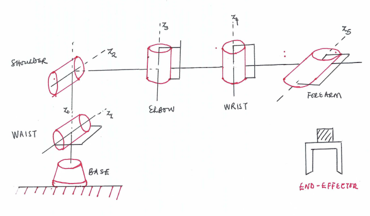
Four rigid bodies (links). Three constraints. Four freedoms. Therefore, $\mathfrak{M} = 6(4 - 3 - 1) + 4 = 4$

Mobility Analysis of The Universal Robot

©Universal Robots A/S, DK.

The Revolute Arm

Falls under so-called RRR kinematic arrangements. Also called a revolute, elbow, or anthorpomorphic manipulator.

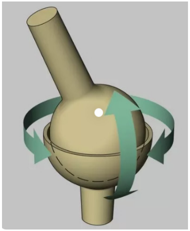


$n = 6$; $k = 5$; $f = 6$

$\therefore \mathfrak{M} = 6(n - k - 1) + \sum f_i$

$\implies 6(6 - 5 - 1) + 6$ or $\mathfrak{M} = 6$.

Mobility of The Stewart-Gough Platform



Mobility of The Stewart-Gough Platform

Unconstrained bodies, n

There are six universal joints that connect the base platform to the prismatic linear actuators. There are six spherical joints that connect the top platform to the top of the prismatic actuators. Altogether, there are $n = 6 + 6 + 2$ or 14 unconstrained rigid bodies.

Constraints, k

Six u-joints. Six spherical joints. Six prismatic joints. Altogether, there are $f = 6 + 6 + 6 := 18$ constraints.

Freedoms, f

Each u-joints has two freedoms. Each spherical joint has three (rotary) freedoms. Each prismatic joint has one freedom. Altogether, there are $f = 6 \times 2 + 6 \times 3 + 6 \times 1 := 36$ freedoms.

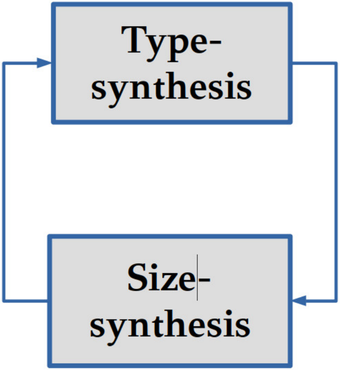
Mechanism Synthesis

Definition (Type-Synthesis)

What type of mechanism is appropriate for a task: A linkage or profile mechanism?

Definition (Size-Synthesis)

What major dimensions of the mechanism is to be synthesized?



Number Synthesis

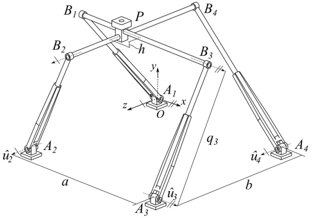
Definition (Number-Synthesis)

That which deals with the freedoms and constraints after the type- and size-synthesis of a mechanism, as well as a kinematic chain's structural analysis is termed the number-synthesis of the mechanism.?

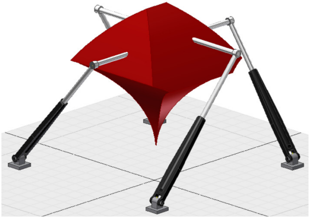
Quiz: Mobility of a Planar Parallel Mechanism

Quiz

Analyze the mobility of the mechanism below.



A planar parallel mechanism. Reprinted from Garcia-Murillo et al.



Workspace of the mechanism.

Quiz: Mechanism Hints

Hint – Mechanism Description

Point $P = (P_x, P_y, P_z)$ is the interconnecting point for all the chains on the mobile platform and the top rods.

Hint – Mechanism Description

The rods that connects points B_1 and B_3, and points B_2 and B_4 are perpendicular. Both rods are connected to the moving platform by prismatic joints, which are separated from each other by a vertical offset h. $\hat{u}_i$ signifies universal joints.

Lecture III Outline

Lecture III Outline

Rigid Body Transformations and Screws Theory.

Rigid body motions: Properties; Direction cosines; Rotation compositions; Rotation Parameterizations.

Rodrigues' formula; the matrix exponential, $SO(3), SO(n), \mathbb{SE}(3)$ group properties.

Transformations: Translations and rotations in $\mathbb{R}^3$, planar rotations, $SO(3), SE(3)$ motions; homogeneous transformations; Euler and Fick angles.

Rigid Body Motions

Rigid Body Motion – Intro

A mapping $g : \mathbb{R}^3 \to \mathbb{R}^3$ is a rigid body motion if

$$\|g(x) - g(y)\| = \|x - y\| \text{ for all } x, y \in \mathbb{R}^3; \tag{6}$$

$$g(x \times y) = g(x) \times g(y) \text{ for all } x, y \in \mathbb{R}^3; \tag{7}$$

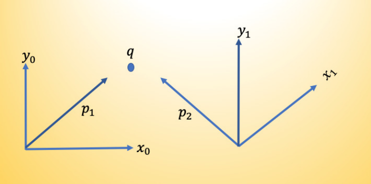
Rigid Body Motion Preserves Inner Products

For two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, $\langle \boldsymbol{a}, \boldsymbol{b} \rangle = g(\boldsymbol{a}) \times g(\boldsymbol{b})$.

Rigid Body Transformations

Translation of Point q between Two Frames

For a reference frame, $o_0x_0y_0$ and a moving coordinate frame, $o_1x_1y_1$, the translation of q is given as below:



$$q^0 = \begin{pmatrix} q_x^0 \\ q_y^0 \end{pmatrix}, \quad q^1 = \begin{pmatrix} q_x^1 \\ q_y^1 \end{pmatrix}$$

Translation of Origin between Two Frames

$$o_1^0 = \begin{pmatrix} o_x^0 \\ o_y^0 \end{pmatrix}, \quad o_0^1 = \begin{pmatrix} o_x^1 \\ o_y^1 \end{pmatrix}. \tag{8}$$

Applications to Screws

Applies to Chasles' displacement theorem and Poinsot's force and couple transformations too.

Screw Transformations

$$\boldsymbol{t}_1^0 = \begin{pmatrix} t_x^0 \\ t_y^0 \end{pmatrix}, \quad \boldsymbol{t}_1^1 = R(-\theta)q^0 \tag{9}$$

$$\boldsymbol{t}_2^0 = R(\theta)q^0, \quad \boldsymbol{t}_2^1 = \begin{pmatrix} t_x^1 \\ t_y^1 \end{pmatrix} \tag{10}$$

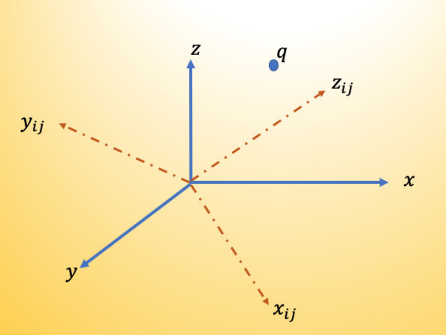
where θ is the angle coordinate frame $o_1x_1y_1$ makes w.r.t $o_0x_0y_0$.

Rotations in $\mathbb{R}^3$

Rotations in $\mathbb{R}^3$

Conventions: Bodies' orientations are measured along a corkscrew direction, specified by a local coordinate frame. Thus, relative orientation is measured from the local coordinate frame to an inertial coordinate frame.

Direction Cosines



Conventions

I: Inertial frame; J: Body frame.
$\boldsymbol{q} : (\boldsymbol{x}_{ij}, \boldsymbol{y}_{ij}, \boldsymbol{z}_{ij}) \in \mathbb{R}^3$: coordinates of the principal axes of J relative to I.

Rotation Matrix from Direction Cosines

Rotation as Composition of Projections Between Frames

$$R_{ij} = \begin{bmatrix} \boldsymbol{x}_{ij} & \boldsymbol{y}_{ij} & \boldsymbol{z}_{ij} \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}. \tag{11}$$

Rotation Matrix as Unit Axes' Dot Products

$$R_{ij} = \begin{bmatrix} \boldsymbol{x}_j \cdot \boldsymbol{x}_i & \boldsymbol{y}_j \cdot \boldsymbol{x}_i & \boldsymbol{z}_j \cdot \boldsymbol{x}_i \\ \boldsymbol{x}_j \cdot \boldsymbol{y}_i & \boldsymbol{y}_j \cdot \boldsymbol{y}_i & \boldsymbol{z}_j \cdot \boldsymbol{y}_i \\ \boldsymbol{x}_j \cdot \boldsymbol{z}_i & \boldsymbol{y}_j \cdot \boldsymbol{z}_i & \boldsymbol{z}_j \cdot \boldsymbol{z}_i \end{bmatrix}. \tag{12}$$

Rotation Matrix from Direction Cosines

Rotation Matrices are Direction Cosines!

$$\boldsymbol{x}_j \cdot \boldsymbol{x}_i = \cos(\measuredangle(\boldsymbol{x}_j, \boldsymbol{x}_i)), \quad \boldsymbol{y}_j \cdot \boldsymbol{x}_i = \cos(\measuredangle(\boldsymbol{y}_j, \boldsymbol{x}_i)), \cdots$$
$$\cdots, \boldsymbol{y}_j \cdot \boldsymbol{z}_i = \cos(\measuredangle(\boldsymbol{y}_j, \boldsymbol{z}_i)), \quad \boldsymbol{z}_j \cdot \boldsymbol{z}_i = \cos(\measuredangle(\boldsymbol{z}_j, \boldsymbol{z}_i)).$$

Properties of Rotation Matrices

Rows of R_{ij} are the unit vector coordinates of I in the frame J so that

$$R_{ij} = R_{ji}^{-1} = R_{ji}^{T}. \tag{13}$$

That is, the inverse of the rotation matrix is equal to its transpose.

Special Orthogonal 3, SO(3)

Orthogonal properties!

Observe: $\det \boldsymbol{R} = \boldsymbol{r}_1^T \cdot (\boldsymbol{r}_2 \times \boldsymbol{r}_3)$. In corkscrew notation, $\det \boldsymbol{R} = +1$ i.e. $\boldsymbol{r}_2 \times \boldsymbol{r}_3 = \boldsymbol{r}_1$ so that $\det \boldsymbol{R} = \boldsymbol{r}_1^T \cdot \boldsymbol{r}_1 = +1$. A matrix that satisfies the above property is said to possess a special orthogonal 3, denoted SO(3), property.

SO(n) Property

Special orthogonal means $\det \boldsymbol{R} = +1$. The set of all SO matrices in $\mathbb{R}^{n \times n}$ is

$$\text{SO(n)} = \{\boldsymbol{R} \in \mathbb{R}^{n \times n} : \boldsymbol{R} \cdot \boldsymbol{R}^T = \boldsymbol{I}, \det \boldsymbol{R} = +1\}. \quad (14)$$

Rotations on Vectors

Rotating a Vector

Suppose that a point p_j is on a frame J, then the vector that connects a point q_j in the frame J to p_j is $v_j = q_j - p_j$. Now, the rotation matrix's action on v_j is

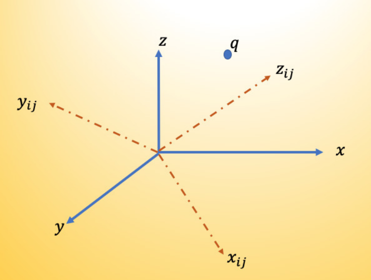
$$\boldsymbol{R}_{ij}(v_j) := \boldsymbol{R}_{ij} q_j - \boldsymbol{R}_{ij} p_j = q_i - p_i = v_i. \quad (15)$$

Planar Rotations

Planar Rotations

Let the angle of rotation between the two coordinate frames be θ. Then,

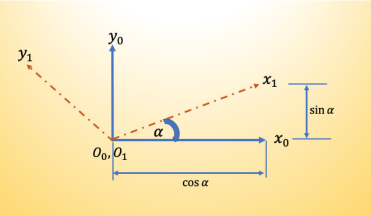
$$R_1^0 = \left(\; x_1^0 \; | \; y_1^0 \; \right) \tag{16}$$



Planar Rotations

It follows that

$$R = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \tag{17}$$



Planar Rotations

Planar Rotations via Direction Cosines

$$R_1^0 = \begin{bmatrix} \boldsymbol{x}_0 \cdot \boldsymbol{x}_1 & \boldsymbol{y}_1 \cdot \boldsymbol{x}_0 \\ \boldsymbol{x}_0 \cdot \boldsymbol{y}_1 & \boldsymbol{y}_1 \cdot \boldsymbol{y}_0 \end{bmatrix} = \begin{bmatrix} cos\alpha & -cos(\pi/2 - \alpha) \\ cos(\pi/2 - \alpha) & cos\alpha \end{bmatrix} = \begin{bmatrix} cos\alpha & -\sin\alpha \\ \sin\alpha & cos\alpha \end{bmatrix}. \tag{18}$$

Projection of y_1 on x_0 is negative because of our adopted right-handed frame.

Axis-Angle Parameterization

Exponential Rotation Coordinates

It can be verified that

$$e^{\tilde{\omega}\theta} = \boldsymbol{I} + \tilde{\omega}\sin\theta + \tilde{\omega}^2(1-\cos\theta), \tag{19}$$

$\tilde{\omega} \in so(3)$ and $so(n) = \{\tilde{\omega} : \tilde{\omega} \in \mathbb{R}^n \times \mathbb{R}^n \mid \tilde{\omega}^T = -\tilde{\omega}^T\}$.

Axis-Angle Parameterization

Parameterization from Exponentiation of Axial Rotation

It can be verified that

$$e^{\tilde{\omega}\theta} = \begin{bmatrix} \omega_x^2 v_\theta + c_\theta & \omega_x\omega_y v_\theta - \omega_z s_\theta & \omega_x\omega_z v_\theta + \omega_y\, s_\theta \\ \omega_x\omega_y v_\theta + \omega_z s_\theta & \omega_y^2 v_\theta + c_\theta & \omega_y\omega_z v_\theta - \omega_x s_\theta \\ \omega_x\omega_z v_\theta - \omega_y s_\theta & \omega_y\omega_z v_\theta + \omega_x s_\theta & \omega_z^2 v_\theta + c_\theta \end{bmatrix}$$

where $v_\theta = 1 - \cos\theta$, $-2\pi n \leq \theta \leq 2\pi n$.

Angle from Exponentiation of Axial Rotation

$$\theta = \cos^{-1} \frac{\text{Trace}(\boldsymbol{R}) - 1}{2} \tag{20}$$

Axis-Angle Parameterization

Axis from Exponentiation of Axial Rotation

Equating the off-diagonal terms, we find that

$$\omega = \frac{1}{2\sin\theta}\begin{pmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{pmatrix} \tag{21}$$

Suppose $\theta \neq 0$, equations (20) together with (21) are the axis-angle representation.

The Skew Symmetrix Matrix

Skew Symmetric Matrix

For a rigid body whose rotation is locally parameterized by $\boldsymbol{R}$ (with $\boldsymbol{R} : \boldsymbol{R}\boldsymbol{R}^T = \boldsymbol{I}$) followed by a translation, $\boldsymbol{d}$.

Compose this transformation as $g = \begin{pmatrix} \boldsymbol{R} & \boldsymbol{I} \\ 0 & 1 \end{pmatrix} \in SE(3)$.

Skew Symmetric Matrix

There exists a skew symmetric matrix, $\tilde{\omega}$, for an axis $\omega = [\omega_x, \omega_y, \omega_z]^T$ about which the rotary and translatory movement occurs such that

$$\tilde{\omega} = \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix} = -\tilde{\omega}^T \qquad (22)$$

Composition of Rotations

Rotations Composition

Let the relative orientation of a frame K to a frame J be $\boldsymbol{R}_{jk}$, and let frame J's relative orientation to frame I be $\boldsymbol{R}_{ij}$, then the relative orientation of frame K w.r.t I is

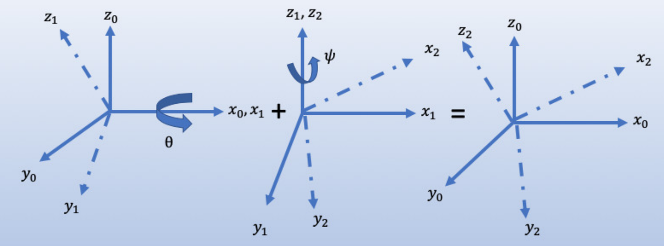
$$\boldsymbol{R}_{ik} = \boldsymbol{R}_{ij} \cdot \boldsymbol{R}_{jk}. \tag{23}$$

Rotations Composition

Equivalent to rotating J relative to frame I according to $\boldsymbol{R}_{ij}$; then aligning frame J to K, we rotate K relative to I according to $\boldsymbol{R}_{jk}$. This frame relative to which rotation occurs is termed the current frame.

Composition of Rotations About A Current Axis

Composition of Rotations About A Current Axis



Compositions

$$\boldsymbol{R} = \boldsymbol{R}_{x,\theta}\boldsymbol{R}_{z,\psi} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_\theta & -s_\theta \\ 0 & s_\theta & c_\theta \end{pmatrix} \cdot \begin{pmatrix} c_\psi & -s_\psi & 0 \\ s_\psi & c_\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{24}$$

$$\boldsymbol{R} = \begin{pmatrix} c_\psi & -s_\psi & 0 \\ 0 & c_\theta c_\psi & -s_\theta \\ s_\theta s_\psi & s_\theta c_\psi & c_\theta \end{pmatrix} \tag{25}$$

Notice how the order of multiplication is carried out, owing to the axis about which we are making the transformation.

Composition of Rotations

Skew matrices and SO(3)

Lemma (Murray, Li, and Sastry, 1994)

The exponential of a skew-matrix $\tilde{\omega}$ that parameterizes a rotation θ about an axis, ω, is in SO(3) $i.e.$, $e^{\tilde{\omega}\theta} \in SO(3)$.

Pre-multiplication of Rotations

A rotation about a fixed axis requires a pre-multiplication.

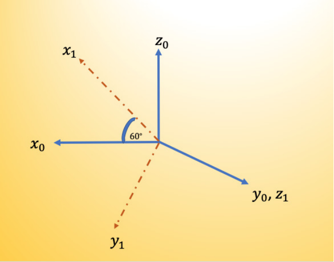
Post-multiplication of Rotations

A rotation about a current axis necessitates a post-multiplication.

Rotations' Composition

Rotations Composition

Suppose all axes of the inertial frame are successively rotated by β around x_0, y_0, z_0 respectively. What is the transformation? Verify that (1) $R_{e,\beta} = I$ where e is the axes about which we are rotating and β is the angle of rotation; (2) The composition of rotations about β and α in a successive manner implies that $R_{z,\beta}, R_{z,\alpha} = R_{z,\beta+\alpha}$, and (3) $(R_{z,\beta})^{-1} = R_{z,-\beta}$.

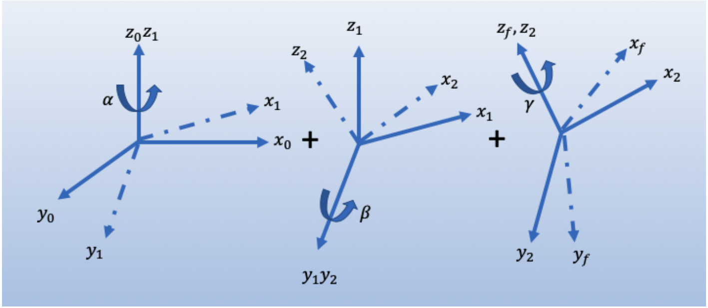


Relative orientation between two frames.

Local Parameterization of Rotations in SO(3)

Theorem (Euler)

Any rotation in $SO(3)$ can be represented as a rotation about a fixed axis $\omega \in \mathbb{R}^3$ through an angle $\theta \in [0, 2\pi)$.



Relative orientation between two frames.

Local Parameterizations of SO(3)

Rotation About Principal x Axis

$$\boldsymbol{R_x}(\theta) = e^{\tilde{\boldsymbol{x}}\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \quad (26)$$

Rotation About Principal y Axis

$$\boldsymbol{R_y}(\phi) = e^{\tilde{\boldsymbol{y}}\phi} = \begin{bmatrix} -\cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix} \quad (27)$$

Local Parameterizations of SO(3)

Rotation About Principal z Axis

$$\boldsymbol{R_z}(\psi) = e^{\tilde{\boldsymbol{z}}\psi} = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{28}$$

Euler Angles as Parameterization of Rotations

"*Read Euler, read Euler, he is the master of us all.*" – Pierre-Simon Laplace.

Euler (ZYZ) Angles

$$\boldsymbol{R}_{ij}(\alpha,\beta,\gamma) = \boldsymbol{R}_z(\alpha)\boldsymbol{R}_y(\beta)\boldsymbol{R}_z(\gamma) \tag{29}$$

$$= \begin{bmatrix} c_\alpha & -s_\alpha & 0 \\ s_\alpha & c_\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix}$$

$$= \begin{bmatrix} c_\alpha c_\beta c_\gamma - s_\alpha s_\gamma & -c_\alpha c_\beta s_\gamma - s_\alpha c_\gamma & c_\alpha s_\beta \\ s_\alpha c_\beta c_\gamma + c_\alpha s_\gamma & -s_\alpha c_\beta s_\gamma + c_\alpha c_\gamma & s_\alpha s_\beta \\ -s_\beta c_\gamma & s_\beta s_\gamma & c_\beta \end{bmatrix} \tag{30}$$

Euler Angles as Parameterization of Rotations

"*The study of Euler's works will remain the best school for the different fields of mathematics, and nothing else can replace it*." – Carl Friedrich Gauss.

Euler (ZYZ) Angles. Case $\sin(\beta) > 0$

Given an R and $\sin(\beta) > 0$, the Euler angles are:

$$\beta = \arctan 2(r_{33}, \sqrt{1 - r_{33}^2}) \tag{31a}$$

$$\alpha = \arctan 2(r_{23}/\sin\beta, r_{13}/\sin\beta) \tag{31b}$$

$$\gamma = \arctan 2(r_{32}/\sin\beta, -r_{31}/\sin\beta) \tag{31c}$$

Euler Angles as Parameterization of Rotations

Euler (ZYZ) Angles. Case $\sin(\beta) < 0$

Given an R and $\sin(\beta) < 0$, the Euler angles are:

$$\beta = \arctan 2(r_{33}, -\sqrt{1 - r_{33}^2}) \tag{32a}$$

$$\alpha = \arctan 2(-r_{23}/ \sin \beta, -r_{13}/ \sin \beta) \tag{32b}$$

$$\gamma = \arctan 2(-r_{32}/ \sin \beta, r_{31}/ \sin \beta) \tag{32c}$$

Euler angles are not unique owing to the sign of the angle about which the y axis rotates!

Euler Angles Drawbacks

Drawbacks

Lack of smooth, global solutions to Euler angles from rotation;

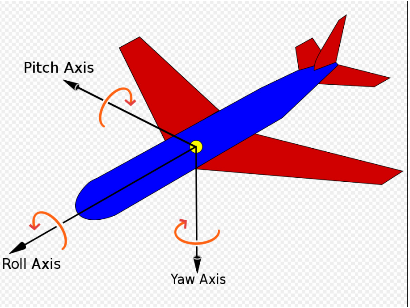
Singularities at $\boldsymbol{R} = \boldsymbol{I}$;

Possible infinite representation of Euler angles at specific configurations e.g. $\boldsymbol{R}(\theta, 0, -\theta) = \boldsymbol{I}$.

Yaw, Pitch, and Roll Axes

Fick angles

Otherwise called the yaw, pitch, and roll angles. $\boldsymbol{R}_{ij}$ found by rotating about the $x-$axis (roll), then a rotation about the $y-$axis (pitch), and finally a rotation about $z-$axis – all in the body frame.



Aircraft Principal Axes in the right-hand frame. Courtesy of Wikimedia commons.

Quaternions

Global Parameterization of Rotations

Instead of locally parameterizing the $SO(3)$ group, quaternions, unlike rotation matrices, globally parameterize the *SO(3)* Lie Group.

Quaternions

Formally, we represent a quaternion as follows:

$$Q = q_0 + q_x \boldsymbol{i} + q_y \boldsymbol{j} + q_z \boldsymbol{k}, \tag{33}$$

where $q_0 \in \mathbb{R}$ is the scalar component of Q and $\boldsymbol{q} = (q_x, q_y, q_z) \in \mathbb{R}^3$ is the vector component.

Quaternions

Quaternions

The unit quaternions are the subset of all $Q \in \mathbb{Q}$ such that $\|Q\| = 1$. For a rotation matrix $R = \exp(\hat{\omega}\theta)$, we have the unit quaternion as

$$Q = (\cos(\theta/2), \omega \sin(\theta/2)), \tag{34}$$

where $\omega \in \mathbb{R}^3$ is the axis of orientation and $\theta \in \mathbb{R}$ is the angle of rotation.

Other Parameterization of Rotations

Fick (ZYX), Helmholtz (YZX) Angles.

We could permute the order of rotation such as rotating successively about different axes. Examples include successive rotations about ZYX axes for the Fick angles and successive rotations about YZX axes for Helmholtz angles.

Fick (ZYX) and Helmholtz (YZX) Angles.

These avoid Euler angle singularities at $\boldsymbol{R} = \boldsymbol{I}$. This does not preclude singularities at other configurations.

Summary of Parameterization of Rotations

Summary of Parameterizations

Rotation matrices can be parameterized in one of many ways depending on our use case. The common examples of parameterizations are

(1) Axis-Angle representation;

(2) Euler angles (ZYZ) representation;

(3) Fick angles ($i.e.$, ZYX or yaw, pitch and roll) representation;

(4) Helmholtz angles (or YZX) angles representation; and

Lecture IV Outline

Screws Theory and Rigid Body Transformations.

Screws (properly revisited): Chasles' and Poinsot's theorem; Displacement and Force screws; Plücker coordinates.

Wrench; Instantaneous screw axis; Couple; Adjoint maps; Velocity transformations – in Body and Spatial Homogeneous Coordinates.

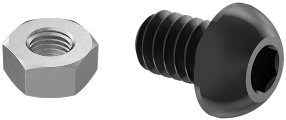
Group theory: The Lie algebra, motions in $\mathfrak{se}(3)$;, and the Lie Group.

Manipulator kinematics: Brockett's exponential map formula. Paden-Kahan subproblems.Denavit-Hartenberg Conventions.

Rigid Body Motions as Screw Motions

Rigid Body Motion as a Screw Motion

The motion of a rigid body is precisely the same as if it were attached to the nut of a literal mechanical screw. Associated with the screw is its pitch.



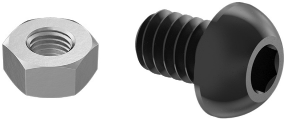
Definition (Screw)

That straight line with which a definite linear magnitude is associated is the unique instantaneous screw axis (ISA), $\mathscr{S}$.

Screw as a Geometric Quantity

Pitch of a Screw

The rectilinear distance, d, through which (a literal nut) nut is translated parallel to the axis of a screw, while the nut is rotated through the angular unit of circular measure, θ, is termed the pitch, h. We say $h\theta = d$.



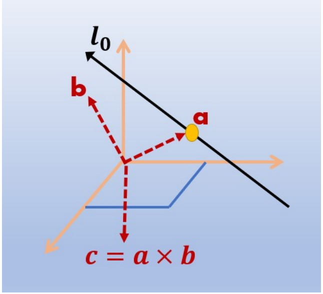
Definition (Screw Coordinates)

Six-vector, $\boldsymbol{s}$, related to the Plücker coordinates, parameterize a screw i.e. $\boldsymbol{s} = (s_1, s_2, s_3, s_4, s_5, s_6)$.

Screw in Plücker Coordinates

Plücker Coordinates

Let $\boldsymbol{a}$ be a point on line $\boldsymbol{\ell}_0$. Let $\boldsymbol{a}$'s direction cosine vector (to be introduced shortly) be $\boldsymbol{b}$. Then, its binormal (moment) vector is $\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b}$. We say the pair $(\boldsymbol{b}, \boldsymbol{c})$ is the Plücker Coordinates of point $\boldsymbol{a}$ on axis $\boldsymbol{\ell}_0$.



Direction Cosines

SLet $| b |= \sqrt{b_1^2 + b_2^2 + b_3^2}$. Then $(\boldsymbol{b}/ \mid b \mid, \boldsymbol{c}/ \mid b \mid)$ are respectively the direction cosines of the line, l_0 and its moment.

Screws and Plücker Coordinates

Screw axis and Plücker Coordinates

$$b_1 = s_1, \quad b_2 = s_2, \quad b_3 = s_3; \tag{35}$$

$$c_1 = s_4 - h \cdot s_1, \quad c_2 = s_5 - h \cdot s_2, \quad c_3 = s_6 - h \cdot s_3. \tag{36}$$

Pitch (of Displacement) in Plücker Coordinates

$$p = \frac{s_1\, s_4 + s_2\, s_5 + s_3\, s_6}{s_1^2 + s_2^2 + s_3^2}, \tag{37}$$

$$|\, s \,| = \sqrt{s_1^2 + s_2^2 + s_3^2} \quad \text{if } h \neq \infty, \tag{38}$$

$$|\, s \,| = \sqrt{s_4^2 + s_5^2 + s_6^2} \quad \text{if } h = \infty \tag{39}$$

Pitch and Magnitude of the screw

Homogeneous Coordinates!

Plücker Coordinates give six unit parameters of a point on a line. Plücker Coordinates are in homogeneous coordinates!

Twist About a Screw (Axis)

Twist

A body's twist about a screw axis is a uniform (infinitesimal) rotation about the screw followed by a uniform (infinitesimal) translation about an axis parallel to the screw, through a distance that is the product of the pitch and the circular measure of rotation.

Twist

A twist requires six algebraic quantities for its complete specification.

Twist in Plücker Coordinates

Definition (Twist Coordinates)

A six-vector, $\boldsymbol{t}$, related to the Plücker coordinates parameterize a twist vector i.e. $\boldsymbol{t} = [(t_1, t_2, t_3), (t_4, t_5, t_6)]$ or $\boldsymbol{t} = (\boldsymbol{\omega}, \boldsymbol{v})$, where $\boldsymbol{\omega} = (t_1, t_2, t_3)$ and $\boldsymbol{v} = (t_4, t_5, t_6)$.

Plücker Coordinates of a Twist

$$b_1 = t_1, \quad b_2 = t_2, \quad b_3 = t_3 \tag{40}$$

$$c_1 = t_4 - p \cdot s_1, \quad c_2 = t_5 - p \cdot s_2, \quad c_3 = t_6 - p \cdot s_3. \tag{41}$$

Twists in Plücker Coordinates

Pitch of the Twist

$$p_t = \frac{t_1\, t_4 + t_2\, t_5 + t_3\, t_6}{t_1^2 + t_2^2 + t_3^2} = \frac{\boldsymbol{\omega} \cdot \boldsymbol{v}}{\boldsymbol{\omega} \cdot \boldsymbol{\omega}}.$$

Pitch of the Twist

Expressed as a ratio of the magnitude of the velocity of a point on the twist axis to the magnitude of the angular velocity about the twist axis.

Twists and Fixed Movements

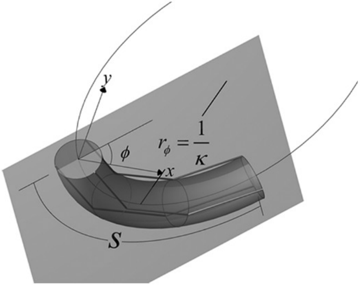
Pure Rotation

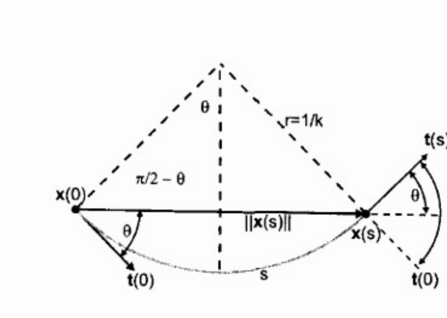
Let pitch be zero. That which results is but pure rotation.

Pure Translation

Let pitch be infinite. That which results cannot be a finite twist, except the amplitude be zero, whereupon the twist becomes a pure translation parallel to the screw.

Curvilinear Displacement: Serret-Frenet Frame





Elephant Trunk Multi-sectional Continuum Model (left), and its Representation in the Serret-Frenet Frame (right).

Plücker Coordinates Example

Chasles' Theorem Applied to The Serret-Frenet Frame

Consider a spatial curve $\boldsymbol{S}$ on the elephant continuum trunk shown earlier. Suppose $\boldsymbol{S}$ is parameterized by its arc length $\boldsymbol{s} \in [0, 1]$. For a point $\boldsymbol{x} = [x, y, z]^T$ on $\boldsymbol{S}$, the unit tangent vector at s is $\boldsymbol{t}(s) = \boldsymbol{dx}/\boldsymbol{ds}$.

Differential Kinematics and The Serret-Frenet Frame

Denote by $\boldsymbol{n}$ the principal normal to S at x; then we must have $\boldsymbol{b} = \boldsymbol{t} \times \boldsymbol{n}$ as the binormal. We say $(\boldsymbol{b}, \boldsymbol{n})$ is the Plücker coordinate of the tangent $\boldsymbol{t}$.

Force

Force

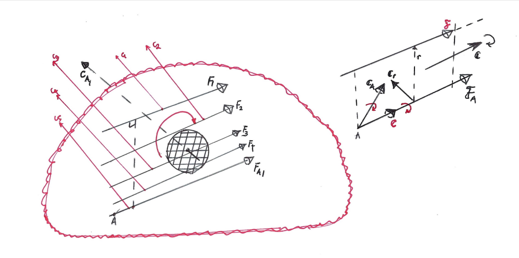
Net force exerted on a body, $\boldsymbol{F} = (f_x, f_y, f_z)$.

Couple of Force

Suppose that $\boldsymbol{F}$ acts along a corkscrew axis. The resulting motion when $\boldsymbol{F}$ makes an infinitesimal rotation about its screw axis is called its couple, $\mathfrak{C} = (c_x, c_y, c_z)$.

Dyname

Dyname: Force and Couple



Description

Forces, F_i, shifted about parallel lines of action yield couples $C_i, i = 1, 2, \cdots$. A dyname $(\mathfrak{F}, \mathfrak{C})$ – introduced by Plücker, 1866 – by a shift of axis, becomes a wrench, $\boldsymbol{w}$ – such that $\mathfrak{F}$ and $\mathfrak{C}$ are parallel to one another.

Complete Wrench on a Screw

Wrench

A wrench requires six algebraic quantities for its complete specification: five ($\{w_i\}_{i=1}^{5}$) specify the screw, the sixth (or its intensity), w_6, specifies the force's magnitude.

Couple's Moment

The moment of the couple is the product of the intensity of the wrench and the and the screw's pitch i.e. $\alpha(\mathfrak{C}) = w_6 \times p_w$.

Wrench on a Screw

Wrench

Simple Definition: A force and a couple both acting in a plane perpendicular to the force.

Definition (Complete Definition)

The resultant canonical system of forces acting on a rigid body, reduced to a resultant force on a point, and acting along the resultant couple that is perpendicular to the plane in which the force acts is called the wrench.

Wrench in Plücker Coordinates

Wrench Coordinates

Wrenches are in the dual space of twists. A six-vector, $\boldsymbol{w}$, related to the Plücker coordinates parameterize a wrench vector i.e. $\boldsymbol{w} = [(w_1, w_2, w_3), (w_4, w_5, w_6)]$ or $\boldsymbol{w} = (\boldsymbol{f}, \boldsymbol{m})$, where $\boldsymbol{f} = (w_1, w_2, w_3)$ and $\boldsymbol{m} = (w_4, w_5, w_6)$.

Plücker Coordinates of a Wrench

$$b_1 = w_1, \quad b_2 = w_2, \quad b_3 = w_3 \tag{42}$$

$$c_1 = w_4 - p \cdot s_1, \quad c_2 = w_5 - p \cdot s_2, \quad c_3 = t_6 - p \cdot w_3. \tag{43}$$

Wrench in Plücker Coordinates

Pitch of the Wrench

$$p_t = \frac{w_1\, w_4 + w_2\, w_5 + w_3\, w_6}{w_1^2 + w_2^2 + w_3^2} = \frac{\boldsymbol{f} \cdot \boldsymbol{m}}{\boldsymbol{f} \cdot \boldsymbol{f}}.$$

Pitch of the Wrench

Expressed as a ratio of the moment applied about a point on the axis to the magnitude of the force applied along the wrench axis.

Wrench's Magnitude

$\|f\| = \sqrt{w_1^2 + w_2^2 + w_3^2}$ if $p_w = 0$ else
$\|m\| = \sqrt{w_4^2 + w_5^2 + w_6^2}$ if $p_w = \infty$.

Wrenches and Fixed Movements

Wrench and Twists

Wrenches are elements of the dual to the Lie algebra—linear functional on the twists. $\boldsymbol{w}^T\boldsymbol{t} = \boldsymbol{F}\cdot\boldsymbol{v} + \mathfrak{C}\cdot\boldsymbol{m}$ is a scalar, invariant to coordinates changes.

Pure Force

Let pitch be zero. That which results is pure force along its screw axis.

Pure Couple

Let pitch be infinite. That which results cannot be a finite wrench, except the intensity be zero, whereupon the wrench becomes a pure couple in a plane that is perpendicular to the screw.

Statics and Instantaneous Kinematics

Statics and kinematics

Statics	Instantaneous Kinematics
Force, $\boldsymbol{F}$ about n.	Infinitesimal rotation, $\boldsymbol{\omega}$
Couple, $\mathfrak{C}$: $[\boldsymbol{F}] \times [\ell]$	Infinitesimal translation, $\boldsymbol{t}$
$p_w = \pm \mathfrak{C}/\boldsymbol{F}$	Pitch of a Wrench, $\boldsymbol{w}$
$\mid \boldsymbol{F} \mid$	Intensity of Wrench

Dyname: $(\boldsymbol{F}, \mathfrak{C})$. Credits: Plücker (1866), Routh (1892).

Plücker Coordinates Kinetics Quiz

Poinsot's Theorem Quiz on a Force and its Moment

Suppose that a force $\boldsymbol{F}$ acts at the point $\boldsymbol{a}$ in the image of Frame 120. What are the Plücker coordinates of the line of force?

Plücker Coordinates Kinetics Quiz

Poinsot's Theorem Quiz on a Force and its Moment

Suppose that a force $\boldsymbol{F}$ acts at the point $\boldsymbol{a}$ in the image of Frame 120. What are the Plücker coordinates of the line of force?

Group Theory Review

The Euclidean Motion

Let $\mathbb{E}^3$ denote the ordinary Cartesian 3-space that admits the standard inner product

$$\langle x, y \rangle = \sum_i x_i \, y_i. \tag{44}$$

Transformations

The set of all length-preserving transformations in $\mathbb{E}^3$ shall be denoted by $\mathbb{E}(3) \in \mathbb{R}^6$ $i.e.$, the family of translations and rotations[a].

[a] Rotations in $\mathbb{E}^3$ are not necessarily proper.

Group Transformation Isomorphism

Brockett, 1990

Euclidean transformation under group composition and Euclidean transformation under group multiplication preserve the isomorphic property.

Example: Affine Euclidean Transformations

We say $\boldsymbol{q}$ defines a Euclidean affine transformation $\boldsymbol{q} = \boldsymbol{R}\boldsymbol{x} + \boldsymbol{d}$ if $\langle \boldsymbol{R}, \boldsymbol{R}^T \rangle = \boldsymbol{I}$ for $(\boldsymbol{q}, \boldsymbol{d}) \in \mathbb{R}^3$. Now, suppose $\boldsymbol{q} = \boldsymbol{R}_1\boldsymbol{x} + \boldsymbol{d}_1$ and $\boldsymbol{p} = \boldsymbol{R}_2\boldsymbol{q} + \boldsymbol{d}_2$, then $\boldsymbol{p} = \boldsymbol{R}_2\boldsymbol{R}_1\boldsymbol{x} + \boldsymbol{d}_2$.

Group Transformation Isomorphism

Example: Euclidean Transformation Identity

$$\begin{pmatrix} \boldsymbol{R}_2 & \boldsymbol{d}_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \boldsymbol{R}_1 & \boldsymbol{d}_1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \boldsymbol{R}_2\boldsymbol{R}_1 & \boldsymbol{R}_2\boldsymbol{d}_1 + \boldsymbol{d}_2 \\ 0 & 1 \end{pmatrix} \tag{45}$$

The isomorphic property (Brockett, 1990)

That matrices of the form (SE(3) matrices): $\begin{pmatrix} \boldsymbol{R} & \boldsymbol{d} \\ 0 & 1 \end{pmatrix}$ are isomorphic.

The General Linear Group

$SO(3)$ as a General Linear Group

The special orthogonal group, $SO(3)$, is a subgroup of the general linear group

$$SO(3) = \{\boldsymbol{R} \in GL(n, \mathbb{R}) : \boldsymbol{R}\,\boldsymbol{R}^T = \boldsymbol{I}, \text{det } \boldsymbol{R} = \boldsymbol{I}\}. \quad (46)$$

The Lie Group

The Lie Group

A group with a topology operation on its set of elements such that the group can be given the structure of a differential manifold with the property that group multiplication and inversion is continuous is called a Lie group.

The Special Euclidean Matrix Group, $SE(3)$

$SE(3)$ is a differentiable manifold, comprised of all the translations and proper rotations that moves a body from one point to another in the ordinary cartesian 3-space $\mathbf{E}^3$.

The Lie Group

The Special Euclidean Matrix Group, $SE(3)$

$$g = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{d} \\ \boldsymbol{0}^T & 1 \end{bmatrix}; g \in SE(3). \tag{47}$$

The Lie Group

The Special Euclidean Matrix Group, $SE(3)$

$$SE(3) = \{(\boldsymbol{R}, d) : \boldsymbol{R} \in SO(3), d \in \mathbb{R}^3\} := SO(3) \times \mathbb{R}^3. \tag{48}$$

I have followed Chasles' notation, who posited that any rigid motion can be formed via a rotation, followed by a translation, and that the rotation and the translation commute i.e. $\boldsymbol{R}\,d = d$.

The Special Euclidean Matrix Group, $SE(3)$

Note: Most authors' notation follow Euclid's theorem i.e. any rigid motion is a translation followed by a rotation about an axis that passes through a pre-specified (fixed) point.

$$SE(3) = \{(d, \boldsymbol{R}) : d \in \mathbb{R}^3, \boldsymbol{R} \in SO(3)\} := \mathbb{R}^3 \times SO(3). \tag{49}$$

The Lie Group

The Special Euclidean Matrix Group, $SE(3)$

Chasles' notation allows for motion representation in form of screw motions.

Commutativity of group operations on $SE(3)$

[Brockett, 1990]: Equation (48) imply that the Lie group is a semidirect product of simple Lie subgroup of orthogonal transformations and the abelian Lie subgroup of all translations.

The Lie Algebra

The Lie Algebra, $\mathfrak{se}(3)$

The Lie algebra is a vector space $\hat{\xi}$ with the antisymmetric bilinear operation $[,] : \hat{\xi} \times \hat{\xi} \to \hat{\xi}$ which satisfies the Jacobi identity,

$$[\hat{\xi}_1, [\hat{\xi}_2, \hat{\xi}_3]] + [\hat{\xi}_2, [\hat{\xi}_3, \hat{\xi}_1]] + [\hat{\xi}_3, [\hat{\xi}_1, \hat{\xi}_2]] = 0. \tag{50}$$

NB: $[,]$ is alternatively the Lie bracket notation with antisymmetry operation $[\hat{\xi}_2, \hat{\xi}_3] = -[\hat{\xi}_3, \hat{\xi}_2]$.

The Lie Algebra Representation

The Lie Algebra Representation, $\mathfrak{se}(3)$

The Lie algebra admits the following homogeneous coordinates representation for a point $q \in \mathbb{R}^3$ on a link that rotates with unit velocity ω,

$$\hat{\boldsymbol{\xi}} = \begin{pmatrix} \tilde{\omega} & v \\ 0 & 0 \end{pmatrix} \in \mathfrak{se}(3), \; \boldsymbol{\xi} = \left(\omega^T, v^T\right)^T \in \mathbb{R}^6 \tag{51}$$

where $v = -\omega \times q$.

The Lie Algebra Representation

The Lie Algebra Representation, $\mathfrak{se}(3)$

Observe:

$$\tilde{\omega} = \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix} \equiv -\tilde{\omega}^T \in \mathfrak{so}(3) \qquad (52)$$

is the skew-symmetric form of the velocity of the tip point, $\omega \in \mathbb{R}^3$.

Lie Algebra Diffeomorphisms

Lie Representation Snippet

Observe:

$$\tilde{(\cdot)}_{SO(3)} : \mathbb{R}^3 \to \mathfrak{so}(3) \tag{53}$$

$$\tilde{(\cdot)}_{SE(3)} : \mathbb{R}^6 \to \mathfrak{se}(3) \tag{54}$$

$\tilde{\omega}(S) \in \mathfrak{se}(3)$: e.g. Twist parameterization of a curve, deformation, screw.
$\omega(S) \in \mathbb{R}^6$: e.g. Motion vector e.g. linear + angular velocities, axial, shear, bending, and torsion motion.

The exponential map belongs to the Lie Group

The exponential map is an element of $SE(3)$

Given $g : \begin{pmatrix} \boldsymbol{R}(\theta) & \boldsymbol{d} \\ 0 & 1 \end{pmatrix} \in SE(3)$ there exists a $\tilde{\boldsymbol{\xi}} = (\tilde{\omega}, v) \in \mathfrak{se}(3)$, such that $exp(\tilde{\boldsymbol{\xi}}\theta) \in SE(3)$[a].

[a] Proof in Murray and Sastry, Prop 2.8.

The exponential map is surjective onto $SE(3)$

Given $g : \begin{pmatrix} \boldsymbol{R}(\theta) & \boldsymbol{d} \\ 0 & 1 \end{pmatrix} \in SE(3)$ there exists a $\begin{pmatrix} \tilde{\omega} & \boldsymbol{d} \\ 0 & 0 \end{pmatrix}; \tilde{\omega} = -\tilde{\omega}^T$, such that $exp(\tilde{\omega}) = g$.

Euler's Rigid Motion Theorem

Surjectivity of g and the Exponential Map

Since

$$\begin{pmatrix} \boldsymbol{R} & \boldsymbol{d} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \boldsymbol{I} & \boldsymbol{d} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \boldsymbol{R} & \boldsymbol{0} \\ 0 & 1 \end{pmatrix} \tag{55a}$$

$$= \begin{pmatrix} \boldsymbol{I} & \boldsymbol{d} \\ 0 & 1 \end{pmatrix} \exp \begin{pmatrix} \tilde{\omega} & \boldsymbol{0} \\ 0 & 0 \end{pmatrix} \tag{55b}$$

Euler's Rigid Motion Theorem

That is, a rigid body motion is a translation, appropriately followed by a rotation about a fixed axis.

Chasles' Rigid Motion Theorem

Surjectivity of g and the Exponential Map

Since

$$\begin{pmatrix} \boldsymbol{R} & \boldsymbol{d} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \boldsymbol{I} & \boldsymbol{c} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \boldsymbol{R} & \boldsymbol{d} \\ 0 & 1 \end{pmatrix} \tag{56a}$$

$$= \begin{pmatrix} \boldsymbol{I} & \boldsymbol{d} \\ 0 & 1 \end{pmatrix} \exp \begin{pmatrix} \tilde{\omega} & \boldsymbol{0} \\ 0 & 0 \end{pmatrix} \tag{56b}$$

Chasles' Rigid Motion Theorem

That is, a rigid body motion is a shift of the origin, followed by a rotation and a translation, $\boldsymbol{d}$, that commute i.e. $\boldsymbol{R}\,\boldsymbol{d} = \boldsymbol{d}$ and $\langle \boldsymbol{c}, \boldsymbol{d} \rangle = 0$.

Screw Motion and Exponential Map

Screw Motion and Exponential Map

Since the range and null spaces of a $\tilde{\omega}$ are orthogonal. Thus,

$$\begin{pmatrix} \boldsymbol{I} & \boldsymbol{c} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \tilde{\omega} & \boldsymbol{d} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{I} & -\boldsymbol{c} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \tilde{\omega} & \boldsymbol{d} - \tilde{\omega}\boldsymbol{c} \\ 0 & 0 \end{pmatrix} \tag{57}$$

establishes that rigid body motions of the form

$$\exp \begin{pmatrix} \tilde{\omega} & \boldsymbol{d} \\ 0 & 0 \end{pmatrix} \theta \tag{58}$$

is essentially a screw motion w.r.t some origin.

See J. M. Hervé, "Analyse strueturelle des mécanismes par groupes de désplacements," Mechanisms and Machine Theory, Vol. 13 (1978) 437-450.

Group Composition and Screws Connection

The Lie Algebra Representation, $\mathfrak{se}(3)$

Observe:

$$\tilde{(\cdot)}_{SO(3)} : \mathbb{R}^3 \to \mathfrak{so}(3) \tag{59}$$

$$\tilde{(\cdot)}_{SE(3)} : \mathbb{R}^6 \to \mathfrak{se}(3) \tag{60}$$

$\tilde{\omega}(S) \in \mathfrak{se}(3)$: Twist parameterization of a curve, deformation, screw.
$\omega(S) \in \mathbb{R}^6$: Motion vector e.g. linear + angular velocities, axial, shear, bending, and torsion motion.