A short treatise on robots' kinematic geometry and kinetics.

Author: Lekan Molu

Dissemination Venue: Microsoft Research RL Group,
New York City, NY 10012

June 09, 2022

Table of Contents I

Lecture IV Outline

Lecture IV Outline

Screws Theory and Rigid Body Transformations.

Screws (properly revisited): Chasles' and Poinsot's theorem; Displacement and Force screws; Plücker coordinates.

Wrench; Instantaneous screw axis; Couple; Adjoint maps; Velocity transformations – in Body and Spatial Homogeneous Coordinates.

Group theory: The Lie algebra, motions in $\mathfrak{se}(3)$;, and the Lie Group.

Manipulator kinematics: Brockett's exponential map formula. Paden-Kahan subproblems.Denavit-Hartenberg Conventions.

Rigid Body Motions as Screws

Rigid Body Motion as a Screw Motion

The motion of a rigid body is precisely the same as if it were attached to the nut of a literal mechanical screw. Associated with the screw is its pitch.

Definition (Screw)

That straight line with which a definite linear magnitude termed the pitch is associated is called the screw.

Screw as a Geometric Quantity

Pitch of a Screw

The rectilinear distance through which (a literal nut) nut is translated parallel to the axis of a screw, while the nut is rotated through the angular unit of circular measure is termed the pitch.

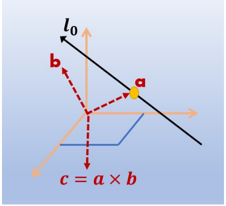
Plücker Coordinates

Let $\boldsymbol{a}$ be a point on line ℓ_0. Let $\boldsymbol{a}$'s direction cosine vector (to be introduced shortly) be $\boldsymbol{b}$. Then, its binormal (moment) vector is $\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b}$. We say the pair $(\boldsymbol{b}, \boldsymbol{c})$ is the Plücker Coordinates of point $\boldsymbol{a}$ on axis ℓ_0.

Screw in Plücker Coordinates

Definition (Screw Coordinates)

Six-vector, $\boldsymbol{s}$, related to the Plücker coordinates, parameterize a screw i.e. $\boldsymbol{s} = (s_1, s_2, s_3, s_4, s_5, s_6)$.



Screws and Plücker Coordinates

Screw axis and Plücker Coordinates

$$b_1 = s_1, \quad b_2 = s_2, \quad b_3 = s_3; \tag{1}$$

$$c_1 = s_4 - p \cdot s_1, \quad c_2 = s_5 - p \cdot s_2, \quad c_3 = s_6 - p \cdot s_3. \tag{2}$$

Pitch in Plücker Coordinates

$$p = \frac{s_1\, s_4 + s_2\, s_5 + s_3\, s_6}{s_1^2 + s_2^2 + s_3^2}, \tag{3}$$

$$|\, s \,| = \sqrt{s_1^2 + s_2^2 + s_3^2} \quad \text{if } p \neq \infty, \tag{4}$$

$$|\, s \,| = \sqrt{s_4^2 + s_5^2 + s_6^2} \quad \text{if } p = \infty \tag{5}$$

Pitch and Magnitude of the screw

Plücker Coordinates' Direction Cosines

Suppose that $h = \sqrt{b_1^2 + b_2^2 + b_3^2}$. Then $(\boldsymbol{b}/h, \boldsymbol{c}/h)$ are respectively the direction cosines of the line, l_0 and its moment.

Homogeneous Coordinates!

Plücker Coordinates give six unit parameters of a point on a line. Plücker Coordinates are in homogeneous coordinates!

Twist About a Screw (Axis)

Twist

A body's twist about s screw is a uniform (infinitesimal) rotation about the screw followed by a uniform (infinitesimal) translation about an axis parallel to the screw, through a distance that is the product of the pitch and the circular measure of rotation.

Twist

A twist requires six s algebraic quantities for its complete specification: five $(\{t_i\}_{i=1}^{5})$ specify the screw, the sixth (or its amplitude) specifies the screw's rotaty angle, t_6.

Twist in Plücker Coordinates

Definition (Twist Coordinates)

A six-vector, $\boldsymbol{t}$, related to the Plücker coordinates parameterize a twist vector i.e. $\boldsymbol{t} = [(t_1, t_2, t_3), (t_4, t_5, t_6)]$ or $\boldsymbol{t} = (\boldsymbol{\omega}, \boldsymbol{v})$, where $\boldsymbol{\omega} = (t_1, t_2, t_3)$ and $\boldsymbol{v} = (t_4, t_5, t_6)$.

Plücker Coordinates of a Twist

$$b_1 = t_1, \quad b_2 = t_2, \quad b_3 = t_3 \tag{6}$$

$$c_1 = t_4 - p \cdot s_1, \quad c_2 = t_5 - p \cdot s_2, \quad c_3 = t_6 - p \cdot s_3. \tag{7}$$

Twists in Plücker Coordinates

Pitch of the Twist

$$p_t = \frac{t_1\, t_4 + t_2\, t_5 + t_3\, t_6}{t_1^2 + t_2^2 + t_3^2} = \frac{\boldsymbol{\omega} \cdot \boldsymbol{v}}{\boldsymbol{\omega} \cdot \boldsymbol{\omega}}.$$

Pitch of the Twist

Expressed as a ratio of the magnitude of the velocity of a point on the twist axis to the magnitude of the angular velocity about the twist axis.

Translation Distance

$d_t = t_6 \times p_t$. The sign expresses the rotation's direction.

Twists and Fixed Movements

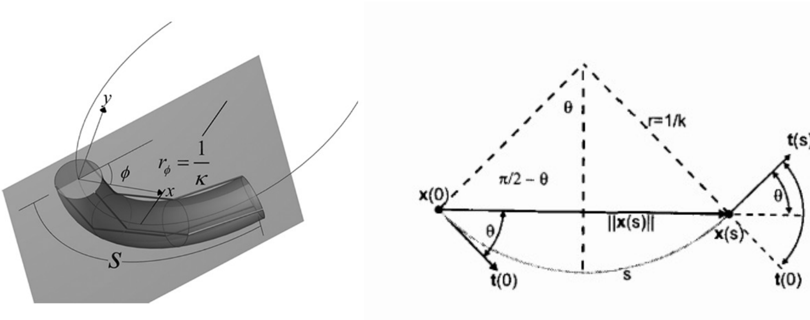
Pure Rotation

Let pitch be zero. That which results is but pure rotation.

Pure Translation

Let pitch be infinite. That which results cannot be a finite twist, except the amplitude be zero, whereupon the twist becomes a pure translation parallel to the screw.

Curvilinear Displacement: Serret-Frenet Frame



Elephant Trunk Multi-sectional Continuum Model (left), and its Representation in the Serret-Frenet Frame.

Plücker Coordinates Example

Chasles' Theorem Applied to The Serret-Frenet Frame

Consider a spatial curve $\boldsymbol{S}$ on the elephant continuum trunk shown earlier. Suppose $\boldsymbol{S}$ is parameterized by its arc length $\boldsymbol{s} \in [0, 1]$. For a point $\boldsymbol{x} = [x, y, z]^T$ on $\boldsymbol{S}$, the unit tangent vector at s is $\boldsymbol{t}(s) = \boldsymbol{dx}/\boldsymbol{ds}$.

Differential Kinematics and The Serret-Frenet Frame

Denote by $\boldsymbol{n}$ the principal normal to $\boldsymbol{S}$ at $\boldsymbol{n}$; then we must have $\boldsymbol{b} = \boldsymbol{t} \times \boldsymbol{n}$ as the binormal. We say $(\boldsymbol{b}, \boldsymbol{n})$ is the Plücker coordinate of the tangent $\boldsymbol{t}$.

Force

Force

Net force exerted on a body, $\boldsymbol{F} = (f_x, f_y, f_z)$.

Couple of Force

Suppose that $\boldsymbol{F}$ acts along a corkscrew axis. The resulting motion when $\boldsymbol{F}$ makes an infinitesimal rotation about its screw axis is called its couple, $\mathfrak{C} = (c_x, c_y, c_z)$.

Complete Wrench on a Screw

Wrench

A wrench requires six s algebraic quantities for its complete specification: five ($\{w_i\}_{i=1}^5$) specify the screw, the sixth (or its intensity), w_6, specifies the force's magnitude.

Couple's Moment

The moment of the couple is the product of the intensity of the wrench and the and the screw's pitch i.e. $\alpha(\mathfrak{C}) = w_6 \times p_w$.

Wrench on a Screw

Wrench

Simple Definition: A force and a couple both acting in a plane perpendicular to the force.

Definition (Complete Definition)

The resultant canonical system of forces acting on a rigid body, reduced to a resultant force on a point, and acting along the resultant couple that is perpendicular to the plane in which the force acts is called the wrench.

Wrench in Plücker Coordinates

Definition (Wrench Coordinates)

A six-vector, $\boldsymbol{w}$, related to the Plücker coordinates parameterize a wrench vector i.e. $\boldsymbol{w} = [(w_1, w_2, w_3), (w_4, w_5, w_6)]$ or $\boldsymbol{w} = (\boldsymbol{f}, \boldsymbol{m})$, where $\boldsymbol{f} = (w_1, w_2, w_3)$ and $\boldsymbol{m} = (w_4, w_5, w_6)$.

Plücker Coordinates of a Wrench

$$b_1 = w_1, \quad b_2 = w_2, \quad b_3 = w_3 \tag{8}$$

$$c_1 = w_4 - p \cdot s_1, \quad c_2 = w_5 - p \cdot s_2, \quad c_3 = t_6 - p \cdot w_3. \tag{9}$$

Wrench in Plücker Coordinates

Pitch of the Wrench

$$p_t = \frac{w_1\, w_4 + w_2\, w_5 + w_3\, w_6}{w_1^2 + w_2^2 + w_3^2} = \frac{\boldsymbol{f} \cdot \boldsymbol{m}}{\boldsymbol{f} \cdot \boldsymbol{f}}.$$

Pitch of the Wrench

Expressed as a ratio of the moment applied about a point on the axis to the magnitude of the force applied along the wrench axis.

Wrench's Magnitude

$$\|f\| = \sqrt{w_1^2 + w_2^2 + w_3^2} \text{ if } p_w = 0 \text{ else}$$
$$\|m\| = \sqrt{w_4^2 + w_5^2 + w_6^2} \text{ if } p_w = \infty.$$

Wrenches and Fixed Movements

Pure Force

Let pitch be zero. That which results is pure force along its screw axis.

Pure Couple

Let pitch be infinite. That which results cannot be a finite wrench, except the intensity be zero, whereupon the wrench becomes a pure couple in a plane that is perpendicular to the screw.

Statics and Instantaneous Kinematics

Statics and kinematics

Statics	Instantaneous Kinematics
Force, $\boldsymbol{F}$ about n.	Infinitesimal rotation, $\boldsymbol{\omega}$
Couple, $\mathfrak{C}$: $[\boldsymbol{F}] \times [\ell]$	Infinitesimal translation, $\boldsymbol{t}$
$p_w = \pm\mathfrak{C}/\boldsymbol{F}$	Pitch of a Wrench, $\boldsymbol{w}$
$\mid \boldsymbol{F} \mid$	Intensity of Wrench

Dyname: $(\boldsymbol{F}, \mathfrak{C})$. Credits: Plücker (1866), Routh (1892).

Plücker Coordinates Kinetics Quiz

Poinsot's Theorem Quiz on a Force and its Moment

Suppose that a force $\boldsymbol{F}$ acts at the point $\boldsymbol{a}$ in the image of Frame 6. What are the Plücker coordinates of the line of force?

Poinsot's Theorem Quiz on a Force and its Moment

Suppose that a force $\boldsymbol{F}$ acts at the point $\boldsymbol{a}$ in the image of Frame 6. What are the Plücker coordinates of the line of force?

Poinsot's Theorem Quiz on a Force and its Moment

Imagine that a force $\boldsymbol{F}$ is acting at the point $\boldsymbol{a}$ in the image of Frame 6. Suppose that $\boldsymbol{\tau}$ is torque acting along the normal to point $\boldsymbol{a}$. Then $(\boldsymbol{f}, \boldsymbol{\tau})$ are the Plücker coordinates of the line of force.

Arithmetics on Screws

Scalar and vector arithmetic operations are valid on infinitesimal screws e.g.

$$c_1 \boldsymbol{s}_1 + c_2 \boldsymbol{s}_2 = 0 \text{ for } c_1,\, c_2 \neq 0 \text{ on screws } \boldsymbol{s}_1, \boldsymbol{s}_2. \tag{10}$$

Plücker Coordinates Kinetics Quiz

Poinsot's Theorem Quiz on a Force and its Moment

Suppose that a force $\boldsymbol{F}$ acts at the point $\boldsymbol{a}$ in the image of Frame 6. What are the Plücker coordinates of the line of force?

Group Theory Review

The Euclidean Motion

Let $\mathbb{E}^3$ denote the ordinary Cartesian 3-space that admits the standard inner product

$$\langle x, y \rangle = \sum_i x_i\, y_i. \tag{11}$$

Transformations

The set of all length-preserving transformations in $\mathbb{E}^3$ shall be denoted by $\mathbb{E}(3) \in \mathbb{R}^6$ $i.e.$, the family of translations and rotations[a].

[a] Rotations in $\mathbb{E}^3$ are not necessarily proper.

Group Transformation Isomorphism

Brockett, 1990

Euclidean transformation under group composition and Euclidean transformation under group multiplication preserve the isomorphic property.

Example: Affine Euclidean Transformations

$\boldsymbol{q}$ defines a Euclidean affine transformation $\boldsymbol{q} = \boldsymbol{R}\boldsymbol{x} + \boldsymbol{d}$ if $\langle \boldsymbol{R}, \boldsymbol{R}^T \rangle = \boldsymbol{I}$ for $(\boldsymbol{q}, \boldsymbol{d}) \in \mathbb{R}^3$. Now, suppose $\boldsymbol{q} = \boldsymbol{R}_1\boldsymbol{x} + \boldsymbol{d}_1$ and $\boldsymbol{p} = \boldsymbol{R}_2\boldsymbol{q} + \boldsymbol{d}_2$, then $\boldsymbol{p} = \boldsymbol{R}_2\boldsymbol{R}_1\boldsymbol{x} + \boldsymbol{d}_2$.

Group Transformation Isomorphism

Example: Euclidean Transformation Identity

$$\begin{pmatrix} \boldsymbol{R}_2 & \boldsymbol{d}_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \boldsymbol{R}_1 & \boldsymbol{d}_1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \boldsymbol{R}_2\boldsymbol{R}_1 & \boldsymbol{R}_2\boldsymbol{d}_1 + \boldsymbol{d}_2 \\ 0 & 1 \end{pmatrix} \tag{12}$$

The isomorphic property (Brockett, 1990)

That matrices of the form (SE(3) matrices): $\begin{pmatrix} \boldsymbol{R} & \boldsymbol{d} \\ 0 & 1 \end{pmatrix}$ are isomorphic.

The General Linear Group

$SO(3)$ as a General Linear Group

The special orthogonal group, $SO(3)$, is a subgroup of the general linear group

$$SO(3) = \{\boldsymbol{R} \in GL(n, \mathbb{R}) : \boldsymbol{R}\,\boldsymbol{R}^T = \boldsymbol{I}, \text{det } \boldsymbol{R} = \boldsymbol{I}\}. \quad (13)$$

The Lie Group

The Lie Group

A group with a topology operation on its set of elements such that the group can be given the structure of a differential manifold with the property that group multiplication and inversion is continuous is called a Lie group.

The Special Euclidean Matrix Group, $SE(3)$

$SE(3)$ is a differentiable manifold, comprised of all the translations and proper rotations that moves a body from one point to another in the ordinary cartesian 3-space $\mathbf{E}^3$.

The Lie Group

The Special Euclidean Matrix Group, $SE(3)$

$$g = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{d} \\ \boldsymbol{0}^T & 1 \end{bmatrix}; g \in SE(3). \tag{14}$$

The Lie Group

The Special Euclidean Matrix Group, $SE(3)$

$$SE(3) = \{(\boldsymbol{R}, d) : \boldsymbol{R} \in SO(3), d \in \mathbb{R}^3\} := SO(3) \times \mathbb{R}^3. \tag{15}$$

I have followed Chasles' notation, who posited that any rigid motion can be formed via a rotation, followed by a translation, and that the rotation and the translation commute i.e. $\boldsymbol{R}\, d = d$.

The Special Euclidean Matrix Group, $SE(3)$

Note: Most authors' notation follow Euclid's theorem i.e. any rigid motion is a translation followed by a rotation about an axis that passes through a pre-specified (fixed) point.

$$SE(3) = \{(d, \boldsymbol{R}) : d \in \mathbb{R}^3, \boldsymbol{R} \in SO(3)\} := \mathbb{R}^3 \times SO(3). \tag{16}$$

The Lie Group

The Special Euclidean Matrix Group, $SE(3)$

Chasles' notation allows for motion representation in form of screw motions.

Commutativity of group operations on $SE(3)$

[Brockett, 1990]: Equation (15) imply that the Lie group is a semidirect product of simple Lie subgroup of orthogonal transformations and the abelian Lie subgroup of all translations.

The Lie Algebra

The Lie Algebra, $\mathfrak{se}(3)$

The Lie algebra is a vector space $\hat{\boldsymbol{\xi}}$ with the antisymmetric bilinear operation $[,] : \hat{\boldsymbol{\xi}} \times \hat{\boldsymbol{\xi}} \to \hat{\boldsymbol{\xi}}$ which satisfies the Jacobi identity,

$$[\hat{\boldsymbol{\xi}}_1, [\hat{\boldsymbol{\xi}}_2, \hat{\boldsymbol{\xi}}_3]] + [\hat{\boldsymbol{\xi}}_2, [\hat{\boldsymbol{\xi}}_3, \hat{\boldsymbol{\xi}}_1]] + [\hat{\boldsymbol{\xi}}_3, [\hat{\boldsymbol{\xi}}_1, \hat{\boldsymbol{\xi}}_2]] = 0. \tag{17}$$

NB: $[,]$ is alternatively the Lie bracket notation with antisymmetry operation $[\hat{\boldsymbol{\xi}}_2, \hat{\boldsymbol{\xi}}_3] = -[\hat{\boldsymbol{\xi}}_3, \hat{\boldsymbol{\xi}}_2]$.

The Lie Algebra Representation

The Lie Algebra Representation, $\mathfrak{se}(3)$

The Lie algebra admits the following homogeneous coordinates representation for a point $q \in \mathbb{R}^3$ on a link that rotates with unit velocity ω,

$$\hat{\boldsymbol{\xi}} = \begin{pmatrix} \tilde{\omega} & v \\ 0 & 0 \end{pmatrix} \in \mathfrak{se}(3), \; \boldsymbol{\xi} = \left(\omega^T, v^T\right)^T \in \mathbb{R}^6 \tag{18}$$

where $v = -\omega \times q$.

The Lie Algebra Representation

The Lie Algebra Representation, $\mathfrak{se}(3)$

Observe:

$$\tilde{\omega} = \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix} \equiv -\tilde{\omega}^T \in \mathfrak{so}(3) \tag{19}$$

is the skew-symmetric form of the velocity of the tip point, $\omega \in \mathbb{R}^3$.

The Lie Algebra Diffeomorphisms

Lie Representation Snippet

Observe:

$$\tilde{(\cdot)}_{SO(3)} : \mathbb{R}^3 \to \mathfrak{so}(3) \quad (20)$$

$$\tilde{(\cdot)}_{SE(3)} : \mathbb{R}^6 \to \mathfrak{se}(3) \quad (21)$$

$\tilde{\omega}(S) \in \mathfrak{se}(3)$: e.g. Twist parameterization of a curve, deformation, screw.
$\omega(S) \in \mathbb{R}^6$: e.g. Motion vector e.g. linear + angular velocities, axial, shear, bending, and torsion motion.

The exponential map belongs to the Lie Group

The exponential map, $exp(\mathfrak{se}(3))$, is an element of $SE(3)$

Given $g : \begin{pmatrix} \boldsymbol{R}(\theta) & \boldsymbol{d} \\ 0 & 1 \end{pmatrix} \in SE(3)$ there exists a $\tilde{\boldsymbol{\xi}} = (\tilde{\omega}, v) \in \mathfrak{se}(3)$, such that $exp(\tilde{\boldsymbol{\xi}}\theta) \in SE(3)$[a].

[a] Proof in Murray and Sastry, Prop 2.8.

The exponential map, $exp(\mathfrak{se}(3))$, is surjective onto $SE(3)$

Given $g : \begin{pmatrix} \boldsymbol{R}(\theta) & \boldsymbol{d} \\ 0 & 1 \end{pmatrix} \in SE(3)$ there exists a $\begin{pmatrix} \tilde{\omega} & \boldsymbol{d} \\ 0 & 0 \end{pmatrix}; \tilde{\omega} = -\tilde{\omega}^T$, such that $exp(\tilde{\omega}) = g$[a].

[a] Proof in Murray and Sastry, Prop 2.9.

Chasles and Affine Transformations

Chasles Theorem and Affine Transformations

$$\begin{pmatrix} \boldsymbol{R} & \boldsymbol{d} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \boldsymbol{I} & \boldsymbol{c} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \boldsymbol{R} & \boldsymbol{d} \\ 0 & 1 \end{pmatrix} \tag{22}$$

with $\boldsymbol{R}\boldsymbol{d} = \boldsymbol{d}$. Note $\langle \boldsymbol{c}, \boldsymbol{d} \rangle = 0$ for $\boldsymbol{c}$ and $\boldsymbol{d}$ to be unique.

Screm Motion and Exponential Map

Screw Motion and Exponential Map (Brockett, 1990)

Range and null space of a $\tilde{\omega}$ are orthogonal. Thus,

$$\begin{pmatrix} \boldsymbol{I} & \boldsymbol{c} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \tilde{\omega} & \boldsymbol{d} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{I} & -\boldsymbol{c} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \tilde{\omega} & \boldsymbol{d} - \tilde{\omega}\boldsymbol{c} \\ 0 & 0 \end{pmatrix} \tag{23}$$

establishes that every motion of the form $\begin{pmatrix} \tilde{\omega} & \boldsymbol{d} \\ 0 & 0 \end{pmatrix} \theta$ is a screw motion w.r.t some origin.

Group Composition and Screws Connection

The Lie Algebra Representation, $\mathfrak{se}(3)$

Observe:

$$\tilde{(\cdot)}_{SO(3)} : \mathbb{R}^3 \to \mathfrak{so}(3) \tag{24}$$

$$\tilde{(\cdot)}_{SE(3)} : \mathbb{R}^6 \to \mathfrak{se}(3) \tag{25}$$

$\tilde{\omega}(S) \in \mathfrak{se}(3)$: Twist parameterization of a curve, deformation, screw.
$\omega(S) \in \mathbb{R}^6$: Motion vector e.g. linear + angular velocities, axial, shear, bending, and torsion motion.