A short treatise on robots' kinematic geometry and kinetics.

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Rodrigues' formula; the matrix exponential, $SO(3), SO(n), \mathbb{SE}(3)$ group properties.

Transformations: Translations and rotations in $\mathbb{R}^3$, planar rotations, $SO(3), SE(3)$ motions; homogeneous transformations; Euler and Fick angles.

Rigid Body Motions

Rigid Body Motion – Intro

A mapping $g : \mathbb{R}^3 \to \mathbb{R}^3$ is a rigid body motion if

$$\|g(x) - g(y)\| = \|x - y\| \text{ for all } x, y \in \mathbb{R}^3; \tag{1}$$

$$g(x \times y) = g(x) \times g(y) \text{ for all } x, y \in \mathbb{R}^3; \tag{2}$$

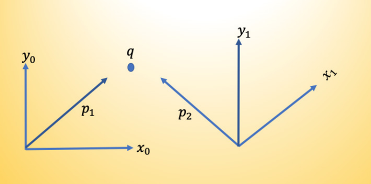
Rigid Body Motion Preserves Inner Products

For two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, $\langle \boldsymbol{a}, \boldsymbol{b} \rangle = g(\boldsymbol{a}) \times g(\boldsymbol{b})$.

Rigid Body Transformations

Translation of Point $\boldsymbol{q}$ between Two Frames

For a reference frame, $o_0 x_0 y_0$ and a moving coordinate frame, $o_1 x_1 y_1$, the translation of $\boldsymbol{q}$ is given as below:



$$q^0 = \begin{pmatrix} q_x^0 \\ q_y^0 \end{pmatrix}, \quad q^1 = \begin{pmatrix} q_x^1 \\ q_y^1 \end{pmatrix}$$

Translation of Origin between Two Frames

$$o_1^0 = \begin{pmatrix} o_x^0 \\ o_y^0 \end{pmatrix}, \quad o_0^1 = \begin{pmatrix} o_x^1 \\ o_y^1 \end{pmatrix}. \tag{3}$$

Rigid Body Transformations

Applications to Screws

Applies to Chasles' displacement theorem and Poinsot's force and couple transformations too.

Screw Transformations

$$\boldsymbol{t}_1^0 = \begin{pmatrix} t_x^0 \\ t_y^0 \end{pmatrix}, \quad \boldsymbol{t}_1^1 = R(-\theta)q^0 \tag{4}$$

$$\boldsymbol{t}_2^0 = R(\theta)q^0, \quad \boldsymbol{t}_2^1 = \begin{pmatrix} t_x^1 \\ t_y^1 \end{pmatrix} \tag{5}$$

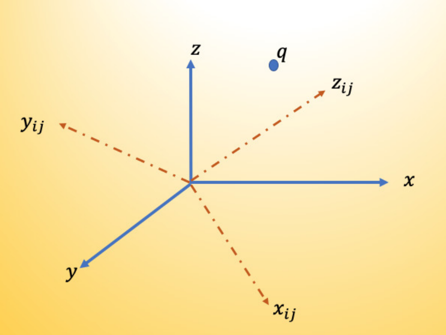
where θ is the angle coordinate frame $o_1x_1y_1$ makes w.r.t $o_0x_0y_0$.

Rotations in $\mathbb{R}^3$

Rotations in $\mathbb{R}^3$

Conventions: Bodies' orientations are measured along a corkscrew direction, specified by a local coordinate frame. Thus, relative orientation is measured from the local coordinate frame to an inertial coordinate frame.

Direction Cosines



Conventions

I: Inertial frame; J: Body frame.

$\boldsymbol{q} : (\boldsymbol{x}_{ij}, \boldsymbol{y}_{ij}, \boldsymbol{z}_{ij}) \in \mathbb{R}^3$: coordinates of the principal axes of J relative to I.

Rotation Matrix from Direction Cosines

Rotation as Composition of Projections Between Frames

$$R_{ij} = \begin{bmatrix} \boldsymbol{x}_{ij} & \boldsymbol{y}_{ij} & \boldsymbol{z}_{ij} \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}. \tag{6}$$

Rotation Matrix as Unit Axes' Dot Products

$$R_{ij} = \begin{bmatrix} \boldsymbol{x}_j \cdot \boldsymbol{x}_i & \boldsymbol{y}_j \cdot \boldsymbol{x}_i & \boldsymbol{z}_j \cdot \boldsymbol{x}_i \\ \boldsymbol{x}_j \cdot \boldsymbol{y}_i & \boldsymbol{y}_j \cdot \boldsymbol{y}_i & \boldsymbol{z}_j \cdot \boldsymbol{y}_i \\ \boldsymbol{x}_j \cdot \boldsymbol{z}_i & \boldsymbol{y}_j \cdot \boldsymbol{z}_i & \boldsymbol{z}_j \cdot \boldsymbol{z}_i \end{bmatrix}. \tag{7}$$

Rotation Matrix from Direction Cosines

Rotation Matrices are Direction Cosines!

$$\boldsymbol{x}_j \cdot \boldsymbol{x}_i = \cos(\measuredangle(\boldsymbol{x}_j, \boldsymbol{x}_i)), \quad \boldsymbol{y}_j \cdot \boldsymbol{x}_i = \cos(\measuredangle(\boldsymbol{y}_j, \boldsymbol{x}_i)), \cdots$$
$$\cdots, \boldsymbol{y}_j \cdot \boldsymbol{z}_i = \cos(\measuredangle(\boldsymbol{y}_j, \boldsymbol{z}_i)), \quad \boldsymbol{z}_j \cdot \boldsymbol{z}_i = \cos(\measuredangle(\boldsymbol{z}_j, \boldsymbol{z}_i)).$$

Properties of Rotation Matrices

Rows of R_{ij} are the unit vector coordinates of I in the frame J so that

$$R_{ij} = R_{ji}^{-1} = R_{ji}^T. \tag{8}$$

That is, the inverse of the rotation matrix is equal to its transpose.

Special Orthogonal 3, SO(3)

Orthogonal properties!

Observe: $\det \boldsymbol{R} = \boldsymbol{r}_1^T \cdot (\boldsymbol{r}_2 \times \boldsymbol{r}_3)$. In corkscrew notation, $\det \boldsymbol{R} = +1$ i.e. $\boldsymbol{r}_2 \times \boldsymbol{r}_3 = \boldsymbol{r}_1$ so that $\det \boldsymbol{R} = \boldsymbol{r}_1^T \cdot \boldsymbol{r}_1 = +1$. A matrix that satisfies the above property is said to possess a special orthogonal 3, denoted SO(3), property.

SO(n) Property

Special orthogonal means $\det \boldsymbol{R} = +1$. The set of all SO matrices in $\mathbb{R}^{n \times n}$ is

$$\text{SO(n)} = \{\boldsymbol{R} \in \mathbb{R}^{n \times n} : \boldsymbol{R} \cdot \boldsymbol{R}^T = \boldsymbol{I}, \det \boldsymbol{R} = +1\}. \tag{9}$$

Rotations on Vectors

Rotating a Vector

Suppose that a point p_j is on a frame J, then the vector that connects a point q_j in the frame J to p_j is $v_j = q_j - p_j$. Now, the rotation matrix's action on v_j is

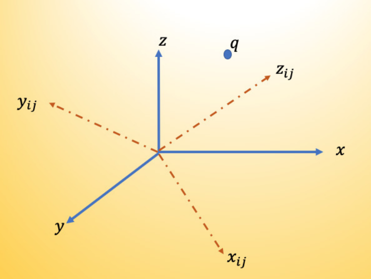
$$\boldsymbol{R}_{ij}(v_j) := \boldsymbol{R}_{ij} q_j - \boldsymbol{R}_{ij} p_j = q_i - p_i = v_i. \quad (10)$$

Planar Rotations

Planar Rotations

Let the angle of rotation between the two coordinate frames be θ. Then,

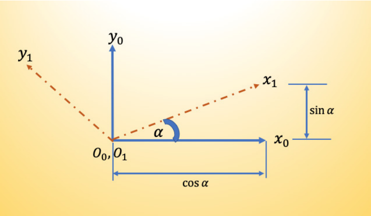
$$R_1^0 = \left(\; x_1^0 \; | \; y_1^0 \; \right) \tag{11}$$



Planar Rotations

It follows that

$$R = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \tag{12}$$



Planar Rotations

Planar Rotations via Direction Cosines

$$R_1^0 = \begin{bmatrix} \boldsymbol{x}_0 \cdot \boldsymbol{x}_1 & \boldsymbol{y}_1 \cdot \boldsymbol{x}_0 \\ \boldsymbol{x}_0 \cdot \boldsymbol{y}_1 & \boldsymbol{y}_1 \cdot \boldsymbol{y}_0 \end{bmatrix} = \begin{bmatrix} cos\alpha & -cos(\pi/2 - \alpha) \\ cos(\pi/2 - \alpha) & cos\alpha \end{bmatrix}$$
$$= \begin{bmatrix} cos\alpha & -\sin\alpha \\ \sin\alpha & cos\alpha \end{bmatrix}. \qquad (13)$$

Projection of y_1 on x_0 is negative because of our adopted right-handed frame.

Composition of Rotations

Rotations Composition

Let the relative orientation of a frame K to a frame J be $\boldsymbol{R}_{jk}$, and let frame J's relative orientation to frame I be $\boldsymbol{R}_{ij}$, then the relative orientation of frame K w.r.t I is

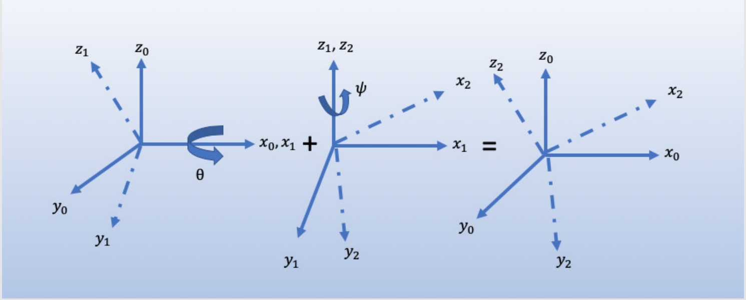
$$\boldsymbol{R}_{ik} = \boldsymbol{R}_{ij} \cdot \boldsymbol{R}_{jk}. \tag{14}$$

Rotations Composition

Equivalent to rotating J relative to frame I according to $\boldsymbol{R}_{ij}$; then aligning frame J to K, we rotate K relative to I according to $\boldsymbol{R}_{jk}$. This frame relative to which rotation occurs is termed the current frame.

Composition of Rotations About A Current Axis

Composition of Rotations About A Current Axis



Composition of Rotations About A Current Axis

Compositions

$$\boldsymbol{R} = \boldsymbol{R}_{x,\theta}\boldsymbol{R}_{z,\psi} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_\theta & -s_\theta \\ 0 & s_\theta & c_\theta \end{pmatrix} \cdot \begin{pmatrix} c_\psi & -s_\psi & 0 \\ s_\psi & c_\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{15}$$

$$\boldsymbol{R} = \begin{pmatrix} c_\psi & -s_\psi & 0 \\ 0 & c_\theta c_\psi & -s_\theta \\ s_\theta s_\psi & s_\theta c_\psi & c_\theta \end{pmatrix} \tag{16}$$

Notice how the order of multiplication is carried out, owing to the axis about which we are making the transformation.

Composition of Rotations About A Current Axis

Skew Symmetry Operations

What happens when the order of multiplication is reversed?

Composition of Rotations

Skew Symmetric Matrix

$$(S)^\wedge = \begin{pmatrix} 0 & -s_z & s_y \\ s_z & 0 & -s_x \\ -s_y & s_x & 0 \end{pmatrix} \tag{17}$$

Skew Symmetric Matrix

Observe $s_{ij} = -s_{ji}$ for $i \neq j$ and $s_i i = 0$

Composition of Rotations

Pre-multiplication of Rotations

A rotation about a fixed axis requires a pre-multiplication.

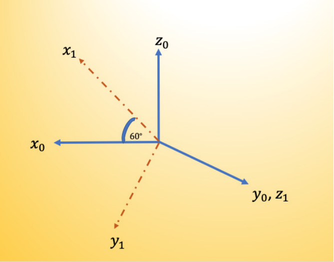
Post-multiplication of Rotations

A rotation about a current axis necessitates a post-multiplication.

Rotations' Composition

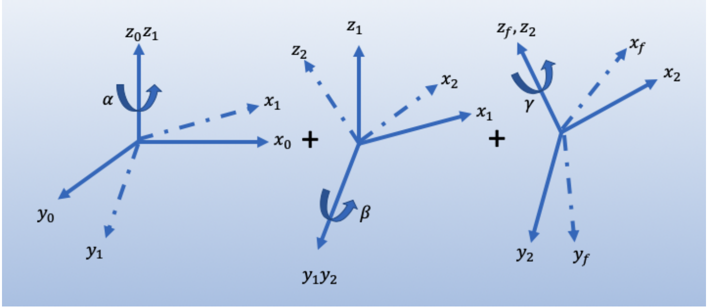
Rotations Composition

Suppose all axes of the inertial frame are successively rotated by β around x_0, y_0, z_0 respectively. What is the transformation? Verify that (1) $R_{e,\beta} = I$ where e is the axes about which we are rotating and β is the angle of rotation; (2) The composition of rotations about β and α in a successive manner implies that $R_{z,\beta}, R_{z,\alpha} = R_{z,\beta+\alpha}$, and (3) $(R_{z,\beta})^{-1} = R_{z,-\beta}$.



Relative orientation between two frames.

Euler Angles as Parameterization of Rotations



Relative orientation between two frames.

Euler Angles as Parameterization of Rotations

Euler (ZYZ) Angles

$$\boldsymbol{R}_{ij}(\alpha, \beta, \gamma) = \boldsymbol{R}_z(\alpha)\boldsymbol{R}_y(\beta)\boldsymbol{R}_z(\gamma) \tag{18}$$

$$= \begin{bmatrix} c_\alpha & -s_\alpha & 0 \\ s_\alpha & c_\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix}$$

$$= \begin{bmatrix} c_\alpha c_\beta c_\gamma - s_\alpha s_\gamma & -c_\alpha c_\beta s_\gamma - s_\alpha c_\gamma & c_\alpha s_\beta \\ s_\alpha c_\beta c_\gamma + c_\alpha s_\gamma & -s_\alpha c_\beta s_\gamma + c_\alpha c_\gamma & s_\alpha s_\beta \\ -s_\beta c_\gamma & s_\beta s_\gamma & c_\beta \end{bmatrix} \tag{19}$$

Euler Angles as Parameterization of Rotations

Euler (ZYZ) Angles. Case $\sin(\beta) > 0$

$$\beta = \arctan 2(r_{33}, \sqrt{1 - r_{33}^2}) \tag{20a}$$

$$\alpha = \arctan 2(r_{23}/\sin\beta, r_{13}/\sin\beta) \tag{20b}$$

$$\gamma = \arctan 2(r_{32}/\sin\beta, -r_{31}/\sin\beta) \tag{20c}$$

where $\arctan 2(y, x)$ determines the quadrant of the angle based on the sign of x and y.

Euler Angles as Parameterization of Rotations

Euler (ZYZ) Angles. Case $\sin(\beta) < 0$

$$\beta = \arctan 2(r_{33}, -\sqrt{1 - r_{33}^2}) \tag{21a}$$

$$\alpha = \arctan 2(-r_{23}/\sin\beta, -r_{13}/\sin\beta) \tag{21b}$$

$$\gamma = \arctan 2(-r_{32}/\sin\beta, r_{31}/\sin\beta) \tag{21c}$$

Euler angles are not unique owing to the sign of the angle about which the y axis rotates!

Other Axes Parameterization of Rotations

Fick (ZYX), Helmholtz (YZX) Angles.

We could permute the order of rotation such as rotating successively about different axes. Examples include successive rotations about ZYX axes for the Fick angles and successive rotations about YZX axes for Helmholtz angles.

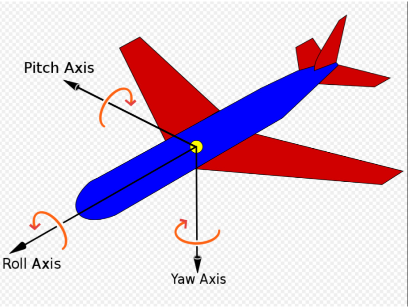
Fick (ZYX) and Helmholtz (YZX) Angles.

These avoid Euler angle singularities at $\boldsymbol{R} = \boldsymbol{I}$. This does not preclude singularities at other configurations.

Fick angles and Yaw, Pitch, and Roll Axes

Fick angles

Otherwise called the yaw, pitch, and roll angles. $\boldsymbol{R}_{ij}$ found by rotating about the $x-$axis (roll), then the $y-$axis (pitch), and finally the $z-$axis – all in the body frame.



Aircraft Principal Axes in the right-hand frame. Courtesy of Wikimedia commons.