Fast Whole-body Strain Regulation in Soft Robots

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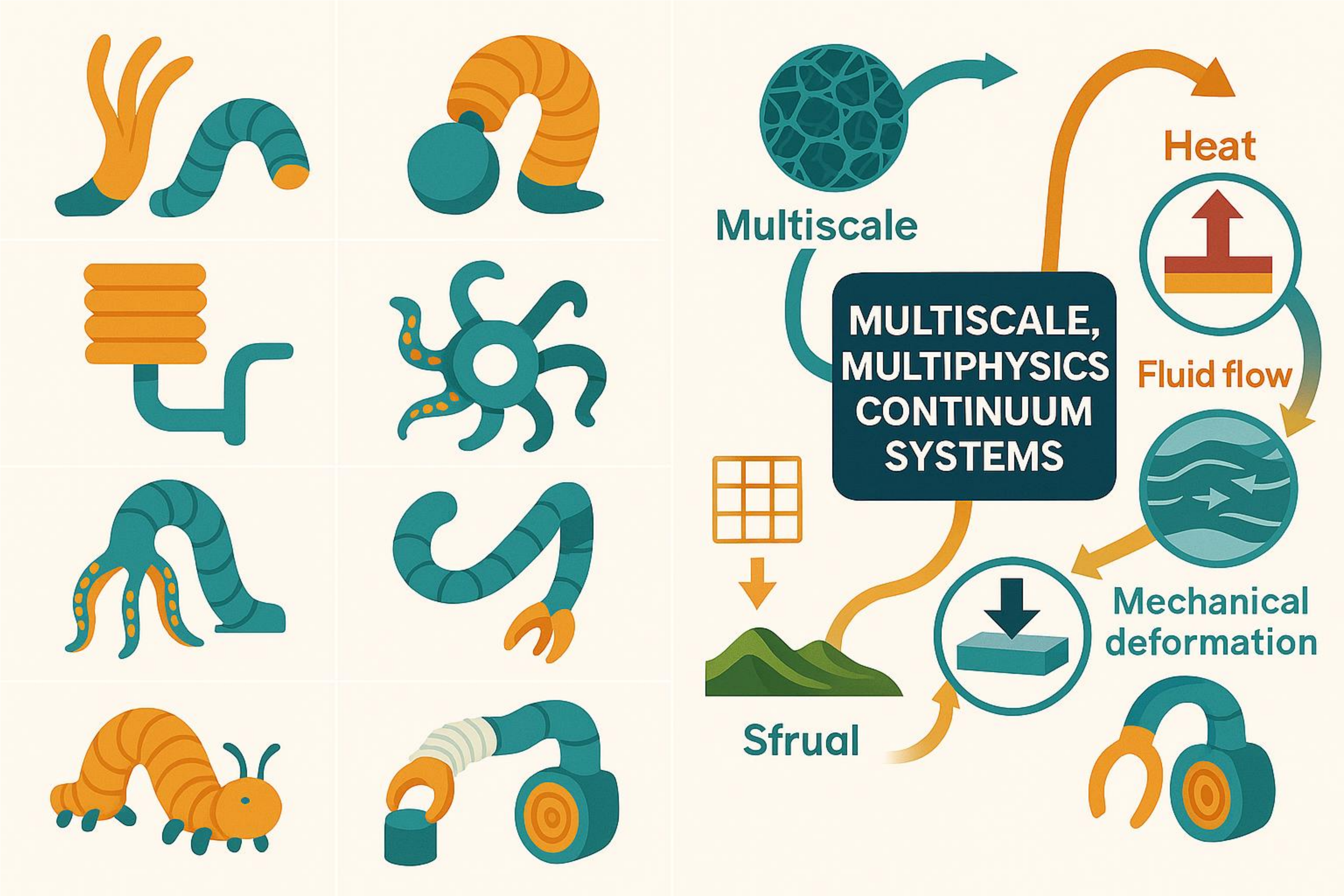
Soft Robotic Systems



Credit: Microsoft CoPilot.

The Piecewise Constant Strain (PCS) Cosserat Model

Octopus robot. Courtesy: IEEE Spectrum

Picture generated by Google Gemini

$$\underbrace{\left[\int_0^{L_N} \boldsymbol{J}^\top \boldsymbol{\mathcal{M}}_a \boldsymbol{J} d\boldsymbol{X}\right]}_{\boldsymbol{M}(\boldsymbol{q})} \ddot{\boldsymbol{q}} + \underbrace{\left[\int_0^{L_N} J^\top \mathrm{ad}^\star_{\boldsymbol{J}\dot{\boldsymbol{q}}} \boldsymbol{\mathcal{M}}_a \boldsymbol{J} d\boldsymbol{X}\right]}_{\boldsymbol{C}_1(\boldsymbol{q},\dot{\boldsymbol{q}})} \dot{\boldsymbol{q}} +$$

$$\underbrace{\left[\int_0^{L_N} \boldsymbol{J}^\top \boldsymbol{\mathcal{M}}_a \dot{\boldsymbol{J}} d\boldsymbol{X}\right]}_{\boldsymbol{C}_2(\boldsymbol{q},\dot{\boldsymbol{q}})} \dot{\boldsymbol{q}} + \underbrace{\left[\int_0^{L_N} \boldsymbol{J}^\top \boldsymbol{\mathcal{D}} \boldsymbol{J} \|\boldsymbol{J}\dot{\boldsymbol{q}}\|_p d\boldsymbol{X}\right]}_{\boldsymbol{D}(\boldsymbol{q},\dot{\boldsymbol{q}})} \dot{\boldsymbol{q}} -$$

$$\underbrace{(1-\rho_f/\rho)\left[\int_0^{L_N} \boldsymbol{J}^\top \boldsymbol{\mathcal{M}} \mathrm{Ad}_{\boldsymbol{g}}^{-1} d\boldsymbol{X}\right]}_{\boldsymbol{N}(q)} \mathrm{Ad}_{\boldsymbol{g}_r}^{-1} \boldsymbol{\mathcal{G}} - \underbrace{\boldsymbol{J}^\top(\bar{\boldsymbol{X}}) \boldsymbol{\mathcal{F}}_p}_{\boldsymbol{F}(\boldsymbol{q})}$$

$$-\underbrace{\int_0^{L_N} \boldsymbol{J}^\top \left[\nabla_x \boldsymbol{\mathcal{F}}_i - \nabla_x \boldsymbol{\mathcal{F}}_a + \mathrm{ad}^\star_{\eta_n} (\boldsymbol{\mathcal{F}}_i - \boldsymbol{\mathcal{F}}_a)\right] d\boldsymbol{X}}_{\boldsymbol{u}(\boldsymbol{q})} = 0,$$

Renda et al. T-RO 2016

$$\boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{z}} + \left[\boldsymbol{C}_1(\boldsymbol{q},\boldsymbol{z}) + \boldsymbol{C}_2(\boldsymbol{q},\boldsymbol{z}) + \boldsymbol{D}(\boldsymbol{q},\boldsymbol{z})\right]\boldsymbol{z} = \tau(\boldsymbol{q}) + \boldsymbol{F}(\boldsymbol{q}) + \boldsymbol{N}(\boldsymbol{q})\mathrm{Ad}_{\boldsymbol{g}_r}^{-1}\mathcal{G}.$$

SoRo's control computational complexity is hard!

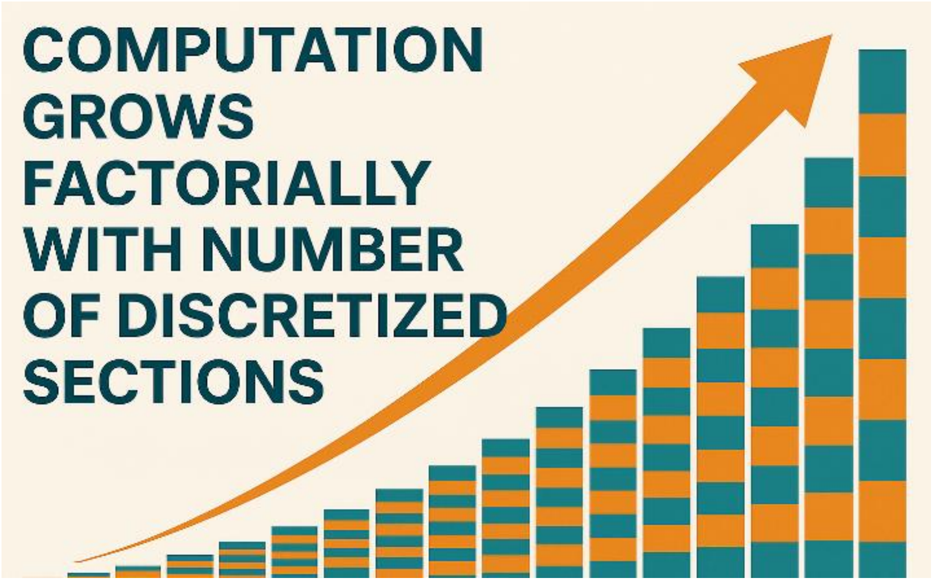
Structural Properties and Control of Soft Robots Modeled as Discrete Cosserat Rods

Lekan Molu and Shaoru Chen

***Abstract*—Soft robots featuring approximate finite-dimensional reduced-order models (undergoing small deformations) are increasingly becoming paramount in literature and applications. In this paper, we consider the piecewise constant strain (PCS) discrete Cosserat model whose dynamics admit the standard Newton-Euler dynamics for a kinetic model. Contrary to popular convention that soft robots under these modeling assumptions admit similar mechanical characteristics to rigid robots, the schemes employed to arrive at the properties for soft robots under finite deformation show a far dissimilarity to those for rigid robots. We set out to first correct the false premise behind this syllogism: from first principles, we established the structural properties of soft slender robots undergoing finite deformation under a discretized PCS assumption; we then utilized these properties to prove the stability of designed proportional-derivative controllers for manipulating the strain states of a prototypical soft robot under finite deformation. Our newly derived results are illustrated by numerical examples on a single arm of the Octopus robot and demonstrate the efficacy of our designed controller based on the derived kinetic properties. This work rectifies previously disseminated kinetic properties of discrete Cosserat-based soft robot models with greater accuracy in proofs and clarity.**

Nonlinear partial differential equations (PDEs) are the standard mathematical machinery for modeling continuum structures with distributed mass. And for soft robots exhibiting infinite degrees-of-freedom (DoF), nonlinear PDEs readily come in handy. However, scanty theory exists for nonlinear PDE analyses. To circumvent the complexity of PDE analyses, researchers have so far exploited approximate finite-dimensional ordinary differential equations (ODEs) [7] for analysis on spatially reduced models.

Tractable reduced-order mathematical models are typically formulated by restricting the range of shapes of the continuum robot to a finite-dimensional functional space over a curve that parameterizes the robot. This is equivalent to taking finite nodal points on the soft robot's body and approximating the dynamics along discretized nodal sections by an ODE. An aggregated ODE of all discretized sections can then be used to model the dynamics of the entire discretized continuum robot. A paramount example is the discrete Cosserat model of Renda et al. [18] whereupon the nonlinear PDE that describes the robot's kinetics in exact form is abstracted to standard Newton-Euler ODEs via



Enter Singularly Perturbed Systems

$$\dot{\boldsymbol{z}}_1 = \boldsymbol{f}(\boldsymbol{z}_1, \boldsymbol{z}_2, \epsilon, \boldsymbol{u}_s, t), \;\; \boldsymbol{z}_1(t_0) = \boldsymbol{z}_1(0), \;\; \boldsymbol{z}_1 \in \mathbb{R}^{6N},$$
$$\epsilon\dot{\boldsymbol{z}}_2 = \boldsymbol{g}(\boldsymbol{z}_1, \boldsymbol{z}_2, \epsilon, \boldsymbol{u}_f, t), \;\; \boldsymbol{z}_2(t_0) = \boldsymbol{z}_2(0), \;\; \boldsymbol{z}_2 \in \mathbb{R}^{6N}$$

General SPT formulation.

Multiphysics, multiscale soft system.

Picture credit: Google Gemini.

$$\dot{\boldsymbol{z}}_1 = \boldsymbol{f}(\boldsymbol{z}_1, \boldsymbol{z}_2, 0, \boldsymbol{u}_s, t), \;\; \boldsymbol{z}_1(t_0) = \boldsymbol{z}_1(0),$$
$$0 = \boldsymbol{g}(\boldsymbol{z}_1, \boldsymbol{z}_2, 0, 0, t).$$

Set ϵ to 0 → Slow subsystem

Assumption 1 (Real and distinct root): Equation (5) has the unique and distinct root $\boldsymbol{z}_2 = \boldsymbol{\phi}(\boldsymbol{z}_1, t)$ (for a sufficiently smooth $\boldsymbol{\phi}(\cdot)$) so that

$$0 = \boldsymbol{g}(\boldsymbol{z}_1, \boldsymbol{\phi}(\boldsymbol{z}_1, t), 0, 0, t) \triangleq \bar{\boldsymbol{g}}(\boldsymbol{z}_1, 0, t), \;\; \boldsymbol{z}_1(t_0) = \boldsymbol{z}_1(0). \quad (6)$$

The slow subsystem therefore becomes

$$\dot{\boldsymbol{z}}_1 = \boldsymbol{f}(\boldsymbol{z}_1, \boldsymbol{\phi}(\boldsymbol{z}_1, t), 0, \boldsymbol{u}_s, t) \triangleq \boldsymbol{f}_s(\boldsymbol{z}_1, \boldsymbol{u}_s, t). \quad (7)$$

$$\frac{d\boldsymbol{z}_1}{dT} = \epsilon\boldsymbol{f}(\boldsymbol{z}_1, \tilde{\boldsymbol{z}}_2 + \boldsymbol{\phi}(\boldsymbol{z}_1, t), \epsilon, \boldsymbol{u}_s, t), \quad (8a)$$
$$\frac{d\tilde{\boldsymbol{z}}_2}{dT} = \epsilon\frac{d\boldsymbol{z}_2}{dt} - \epsilon\frac{\partial\boldsymbol{\phi}}{\partial\boldsymbol{z}_1}\dot{\boldsymbol{z}}_1, \quad (8b)$$
$$= \boldsymbol{g}(\boldsymbol{z}_1, \tilde{\boldsymbol{z}}_2 + \boldsymbol{\phi}(\boldsymbol{z}_1, t), \epsilon, \boldsymbol{u}_f, t) - \epsilon\frac{\partial\boldsymbol{\phi}(\boldsymbol{z}_1, t)}{\partial\boldsymbol{z}_1}\dot{\boldsymbol{z}}_1. \quad (8c)$$

Fast subsystem on time scale: $T = t/\epsilon$

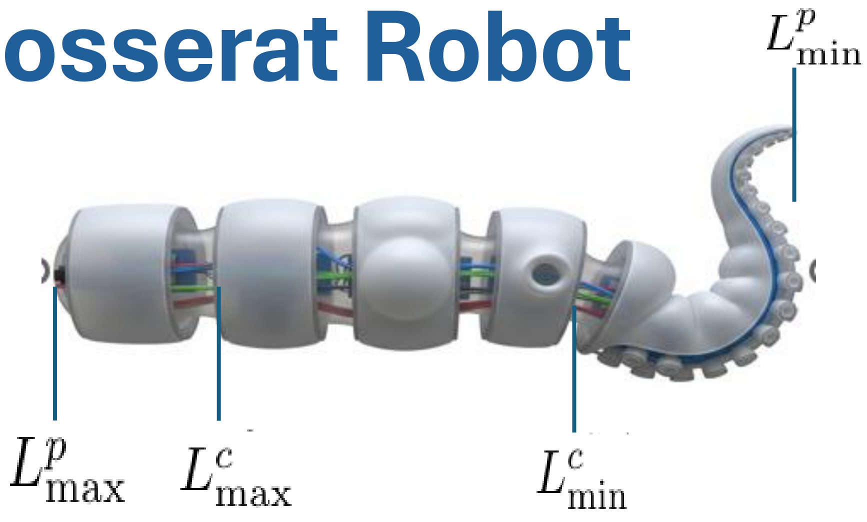
Singularly Perturbed Soft Cosserat Robot

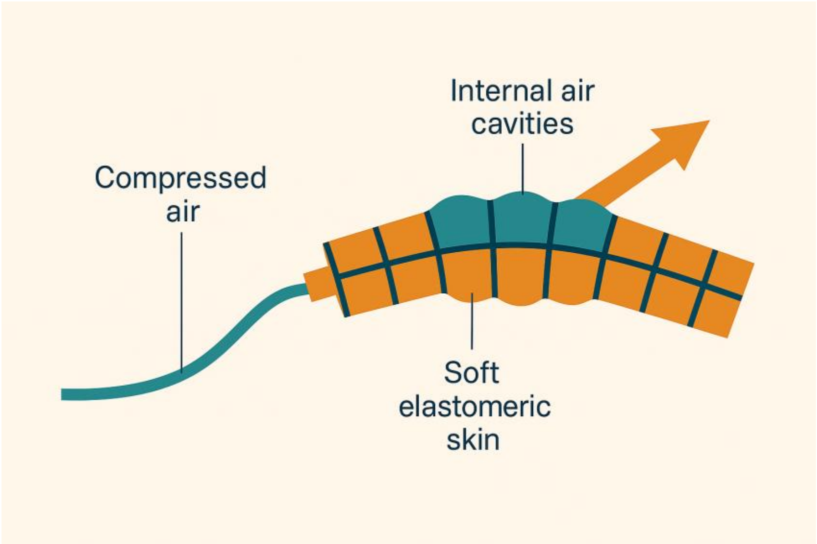
Aggregate the robot's distributed mass, $\mathcal{M}$, inertia into a core active component, $\mathcal{M}_i^{\text{core}}$, and set the passive components as $\mathcal{M}^{\text{pert}} = \mathcal{M} \setminus \mathcal{M}^{\text{core}}$

Then the mass and Coriolis forces adopts the following representation

$$\text{where} \quad \boldsymbol{M}^p = \int_{L^p_{\max}}^{L^p_{\min}} \boldsymbol{J}^\top \boldsymbol{\mathcal{M}}^{pert} \boldsymbol{J} dX$$

$$\boldsymbol{M}(\boldsymbol{q}) = (\boldsymbol{M}^c + \boldsymbol{M}^p)(\boldsymbol{q}),\ \boldsymbol{N} = (\boldsymbol{N}^c + \boldsymbol{N}^p)(\boldsymbol{q}),$$
$$\boldsymbol{F}(\boldsymbol{q}) = (\boldsymbol{F}^c + \boldsymbol{F}^p)(\boldsymbol{q}), \quad \boldsymbol{D}(\boldsymbol{q}) = (\boldsymbol{D}^c + \boldsymbol{D}^p)(\boldsymbol{q})$$
$$\boldsymbol{C}_1(\boldsymbol{q}, \dot{\boldsymbol{q}}) = (\boldsymbol{C}_1^c + \boldsymbol{C}_1^p)(\boldsymbol{q}, \dot{\boldsymbol{q}}),$$
$$\boldsymbol{C}_2(\boldsymbol{q}, \dot{\boldsymbol{q}}) = (\boldsymbol{C}_2^c + \boldsymbol{C}_2^p)(\boldsymbol{q}, \dot{\boldsymbol{q}})$$



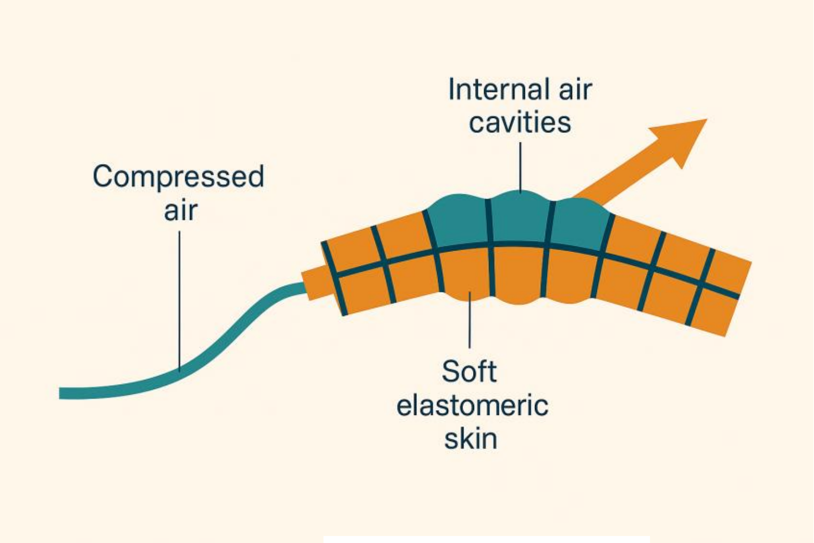


Picture credit: Google Gemini.

Dynamics Separation with Perturbation Parameter

The mass matrix then decomposes as

$$M = \underbrace{\begin{bmatrix} \mathcal{H}_{\text{fast}} & 0 \\ 0 & 0 \end{bmatrix}}_{M^c(q)} + \underbrace{\begin{bmatrix} 0 & \mathcal{H}_{\text{slow}}^{\text{fast}} \\ \mathcal{H}_{\text{slow}}^{\text{fast}}{}^{\top} & \mathcal{H}_{\text{slow}} \end{bmatrix}}_{M^p(q)},$$



$M^c(q)$ and $M^p(q)$ are invertible (Molu & Chen, CDC 2024)

Introducing the perturbation parameter, $\epsilon = \|M^p\|/\|M^c\|$ We may define the matrix, $\bar{M}^p = M^p/\epsilon$

So that we can write,

$$(M^c + \epsilon \bar{M}^p)\dot{z} = s + u,$$

where

$$s = \begin{bmatrix} s_{\text{fast}} \\ s_{\text{slow}} \end{bmatrix} = \begin{bmatrix} F^c + N^c \text{Ad}_{g_r}^{-1} \mathcal{G} - [C_1^c + C_2^c + D^c] z_{\text{fast}} \\ F^p + N^p \text{Ad}_{g_r}^{-1} \mathcal{G} - [C_1^p + C_2^p + D^p] z_{\text{slow}} \end{bmatrix}. \quad (13)$$

Singularly perturbed soft robot form

Suppose that

$$\bar{M}^p = \begin{bmatrix} \bar{M}_{11}^p & \bar{M}_{12}^p \\ \bar{M}_{21}^p & \bar{M}_{22}^p \end{bmatrix} \text{ and } \boldsymbol{\Delta} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \bar{M}_{21}^p \boldsymbol{\mathcal{H}}_{\text{fast}}^{-1} & \mathbf{0} \end{bmatrix},$$

Then, we may write

$$\begin{bmatrix} \boldsymbol{\mathcal{H}}_{\text{fast}} & \bar{M}_{12}^p \\ \mathbf{0} & \bar{M}_{22}^p \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{z}}_{\text{fast}} \\ \epsilon \dot{\boldsymbol{z}}_{\text{slow}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{s}_{\text{fast}} \\ \boldsymbol{s}_{\text{slow}} - \epsilon \bar{M}_{21}^p \boldsymbol{\mathcal{H}}_{\text{fast}}^{-1} \boldsymbol{s}_{\text{fast}} \end{bmatrix} + \begin{bmatrix} \boldsymbol{u}_{\text{fast}} \\ \boldsymbol{u}_{\text{slow}} - \epsilon \bar{M}_{21}^p \boldsymbol{\mathcal{H}}_{\text{fast}}^{-1} \boldsymbol{u}_{\text{fast}} \end{bmatrix} \quad (16)$$

Fast subdynamics extraction

Set $T = t/\epsilon$, with $dT/dt = 1/\epsilon$

Then, $\dot{\boldsymbol{z}}_{\text{fast}} = \frac{d\boldsymbol{z}_{\text{fast}}}{dt} \equiv \frac{1}{\epsilon}\frac{d\boldsymbol{z}_{\text{fast}}}{dT} \triangleq \frac{1}{\epsilon}\boldsymbol{z}'_{\text{fast}}$

and $\epsilon \dot{\boldsymbol{z}}_{\text{slow}} = \boldsymbol{z}'_{\text{slow}}$.

So that,

$$\boldsymbol{z}'_{\text{fast}} = \epsilon \boldsymbol{\mathcal{H}}_{\text{fast}}^{-1}(\boldsymbol{s}_{\text{fast}} + \boldsymbol{u}_{\text{fast}}) - \boldsymbol{\mathcal{H}}_{\text{fast}}^{-1} \boldsymbol{\mathcal{H}}_{\text{slow}}^{\text{fast}} \boldsymbol{z}'_{\text{slow}}$$
$$\boldsymbol{z}'_{\text{slow}} = \boldsymbol{\mathcal{H}}_{\text{slow}}^{-1}(\boldsymbol{s}_{\text{slow}} - \boldsymbol{u}_{\text{slow}}) - \boldsymbol{\mathcal{H}}_{\text{fast}}^{-1}(\boldsymbol{s}_{\text{fast}} - \boldsymbol{u}_{\text{fast}})$$

A backstepping nonlinear multi-scale controller

Theorem 1: The control law

$$\boldsymbol{q}_{\text{fast}}^{d}(t_f) - \boldsymbol{q}_{\text{fast}}(t_f) + \boldsymbol{q}_{\text{fast}}^{\prime d}(t_f)$$

is sufficient to guarantee an exponential stability of the origin of $\boldsymbol{\theta}' = \boldsymbol{\nu}$ such that for all $t_f \geq 0$, $\boldsymbol{q}_{\text{fast}}(t_f) \in S$ for a compact set $S \subset \mathbb{R}^{6N}$. That is, $\boldsymbol{q}_{\text{fast}}(t_f)$ remains bounded as $t_f \to \infty$.

Where,

$$[\boldsymbol{\theta}^\top, \boldsymbol{\phi}^\top]^\top = [\boldsymbol{q}_{\text{fast}}^\top, \boldsymbol{z}_{\text{fast}}^\top]^\top \text{ where } \boldsymbol{\theta}' = \epsilon \boldsymbol{z}_{\text{fast}}.$$

Theorem 2: Under the tracking error $\boldsymbol{e}_2 = \boldsymbol{\phi} - \boldsymbol{\nu}$ and matrices $(\boldsymbol{K}_p, \boldsymbol{K}_q) = (\boldsymbol{K}_p^\top, \boldsymbol{K}_q^\top) > 0$, the control input

$$\boldsymbol{u}_{\text{fast}} = \frac{1}{\epsilon}\boldsymbol{\mathcal{H}}_{\text{fast}}[\boldsymbol{q}_{\text{fast}}^{\prime\prime d} + \boldsymbol{e}_1 - 2\boldsymbol{e}_2 - \boldsymbol{K}_q^\top(\boldsymbol{K}_q\boldsymbol{K}_q^\top)^{-1}\boldsymbol{K}_p\boldsymbol{e}_1] + \frac{1}{\epsilon}\boldsymbol{\mathcal{H}}_{\text{slow}}^{\text{fast}}\boldsymbol{z}_{\text{slow}}' - \boldsymbol{s}_{\text{fast}} \quad (24)$$

exponentially stabilizes the fast subdynamics (18).

Theorem 3: The control law

$$\boldsymbol{u}_{\text{slow}} = \boldsymbol{\mathcal{H}}_{\text{slow}}(\boldsymbol{e}_1 - \boldsymbol{e}_2 - \boldsymbol{e}_3 + \ddot{\boldsymbol{q}}_{\text{fast}}^{d}) - \boldsymbol{s}_{\text{slow}}$$

exponentially stabilizes the slow subdynamics.

A backstepping nonlinear multi-scale controller

4) Stability of the singularly perturbed interconnected system: Let $\varepsilon = (0, 1)$ and consider the composite Lyapunov function candidate $\Sigma(\boldsymbol{z}_{\text{fast}}, \boldsymbol{z}_{\text{slow}})$ as a weighted combination of $\boldsymbol{V}_2$ and $\boldsymbol{V}_3$ i.e. ,

$$\boldsymbol{\Sigma}(\boldsymbol{z}_{\text{fast}}, \boldsymbol{z}_{\text{slow}}) = (1 - \varepsilon)\boldsymbol{V}_2(\boldsymbol{z}_{\text{fast}}) + \varepsilon \boldsymbol{V}_3(\boldsymbol{z}_{\text{slow}}),\ 0 < \varepsilon < 1. \tag{35}$$

It follows that,

$$\begin{aligned}\dot{\boldsymbol{\Sigma}}(\boldsymbol{z}_{\text{fast}}, \boldsymbol{z}_{\text{slow}}) &= (1 - \varepsilon)[\boldsymbol{e}_1^\top \boldsymbol{K}_p \dot{\boldsymbol{e}}_1 + \boldsymbol{e}_2^\top \boldsymbol{K}_q \dot{\boldsymbol{e}}_2] + \varepsilon \boldsymbol{e}_3^\top \boldsymbol{K}_r \dot{\boldsymbol{e}}_3, \\ &= -2(\boldsymbol{V}_2 + \boldsymbol{V}_3) + 2\varepsilon \boldsymbol{V}_2 \leq 0 \end{aligned} \tag{36}$$

which is clearly negative definite for any $\varepsilon \in (0, 1)$. Therefore, we conclude that the origin of the singularly perturbed system is asymptotically stable under the control laws.

$$\boldsymbol{u}(\boldsymbol{z}_{\text{fast}}, \boldsymbol{z}_{\text{slow}}) = (1 - \varepsilon)\boldsymbol{u}_{\text{fast}} + \varepsilon \boldsymbol{u}_{\text{slow}}. \tag{37}$$

Numerical Results

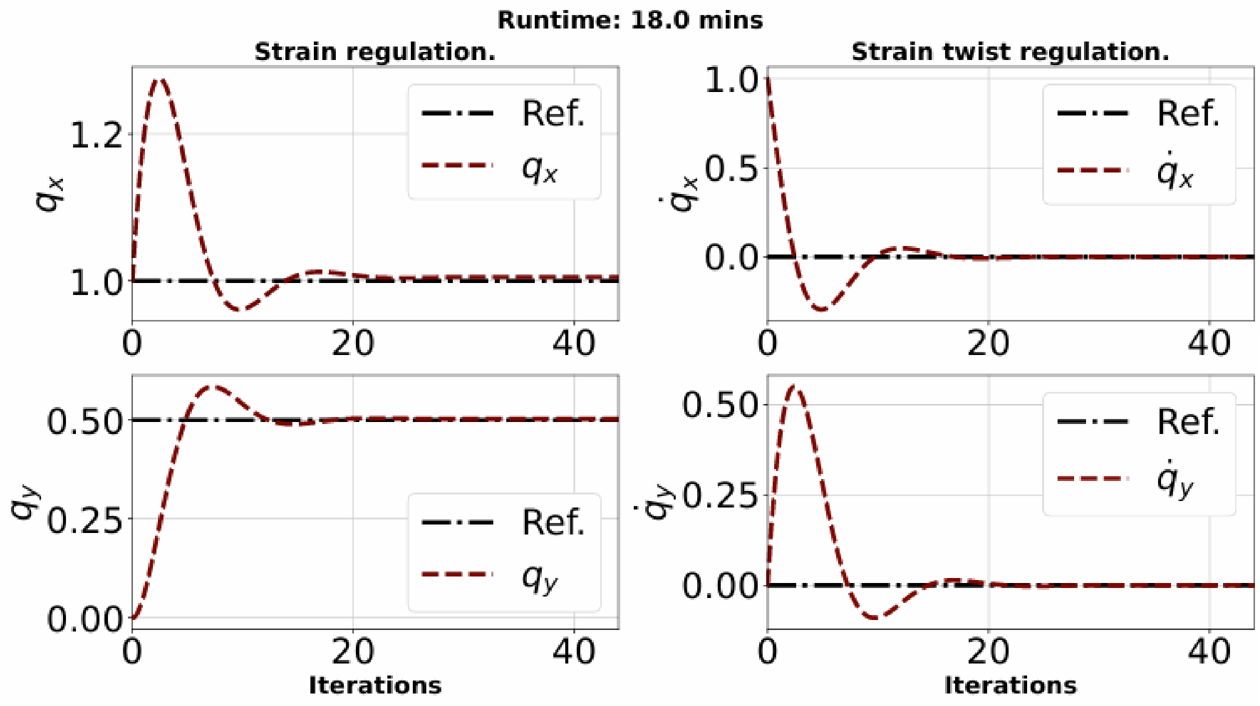


Fig. 2. Backstepping control on the singularly perturbed soft robot system with 10 discretized pieces, divided into 6 fast and 4 slow pieces. For a tip load of $\mathcal{F}_p^y = 10\,N$, the backstepping gains were set as $K_p = 10$, $K_d = 2.0$ for a desired joint configuration $\xi^d = [0, 0, 0, 1, 0.5, 0]^\top$ and $\eta^d = \mathbf{0}_{6\times1}$ that is uniform throughout the robot sections.

Pieces			Runtime (mins)	
Total	Fast	Slow	Hierarchical SPT (mins)	Single-layer PD control (hours)
6	4	2	18.01	51.46
8	5	3	30.87	68.29
10	7	3	32.39	107.43

TABLE I

TIME TO REACH STEADY STATE.

Numerical Results – System Setup

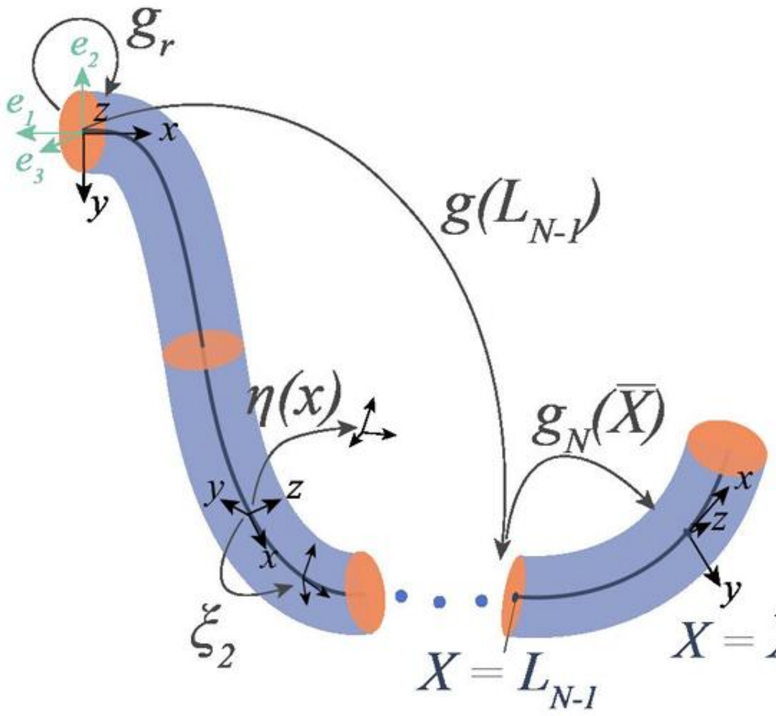


Fig. 1. Simplified configuration of an Octopus arm, reprinted from Molu and Chen [9].

The robot's z-axis is offset in orientation from the inertial frame by -90 deg so that a transformation from the base to inertial frames is

$$\boldsymbol{g}_r = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Tip wrench at $\bar{X} = L$ is ,

$$\boldsymbol{\mathcal{F}}_p = \text{diag}\left(\boldsymbol{R}^\top(L), \boldsymbol{R}^\top(L)\right)\begin{pmatrix} \boldsymbol{0}_{3\times1} & 0 & 10 & 0 \end{pmatrix}^\top$$

Param	Symbol	Value
Reynold's #		0.82
Young's Mod.	**E**	110*kPa*
Shear visc.	**J**	3 *kPa*

Numerical Results – System Setup

Param	Symbol	Value
Reynold's #		0.82
Young's Mod.	**E**	110*kPa*
Shear visc.	**J**	3 *kPa*
Bending 2nd Inertia	$I_y = I_z = \pi r^4/4$	
Torsion 2nd Inert	$I_x = \pi r^4/2$	
Material abscissa	$L = 2m$	
Poisson ratio	ρ	0.45
Mass density	$\mathcal{M} = \rho \cdot \mathrm{diag}([I_x, I_y, I_z, A, A, A])$	
Drag stiffness matrix	$\boldsymbol{D} = -\rho_w \nu^T \nu \breve{\boldsymbol{D}} \nu / \lvert \nu \rvert$	

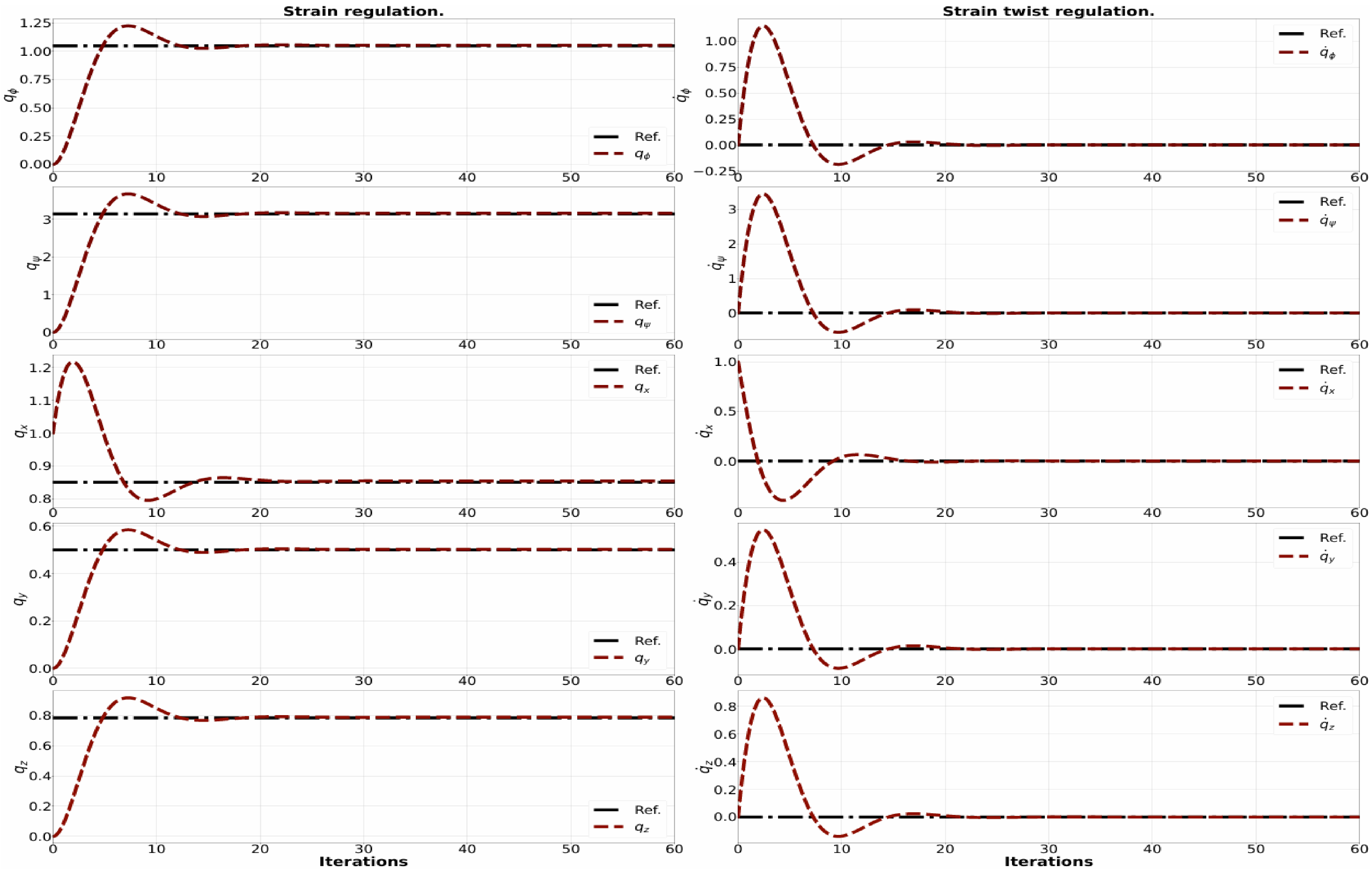


Fig. 3. Backstepping control on the singularly perturbed soft robot system with 10 pieces 4 slow and 6 fast sections.

Conclusion

- Thank you!

- Microsoft Research, NYC.

  - Questions?
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